Subset Simulation: observations and strategies to enhance its performance

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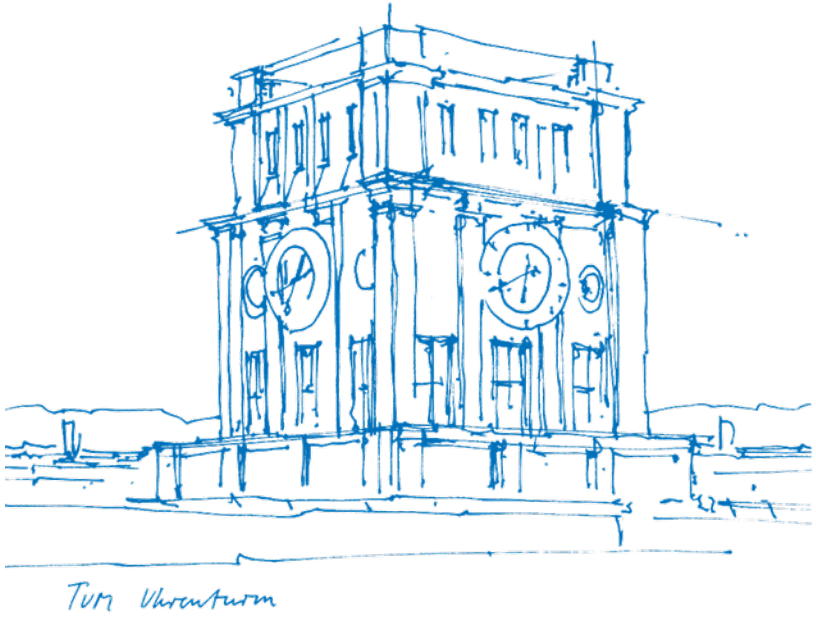


Reliability Problem

Subset Simulation – general idea

Subset Simulation – MCMC sampling

Numerical examples

Reliability Problem

$$P_f = \Pr\left(g(\mathbf{x}) \leq 0\right) = \int_{g(\mathbf{x})\leq 0} p_{\mathbf{X}}(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

P_f Probability of failure

$g(\cdot)$ limit-state function

$\mathbf{x}$ N-dimensional vector of random variables

$p_{\mathbf{X}}(\cdot)$ joint probability density function of $\mathbf{x}$

Transformation to standard Normal space

$$P_f = \int_{G(\mathbf{u}) \leq 0} \varphi_N(\mathbf{u}) \, \mathrm{d}\mathbf{u}$$

$G(\cdot)$ limit-state function in standard Normal space

$\mathbf{u}$ N-dimensional vector of independent standard Normal random variables

$\varphi_N(\cdot)$ probability density function of N independent standard Normal random variables

How-to:

1. generate standard Normal sample $\mathbf{u}$
2. transform $\mathbf{u}$ to $\mathbf{x}$
   - Rosenblatt transformation
   - Nataf transformation
3. evaluate limit-state function $g(\mathbf{x})$

Subset Simulation – general idea

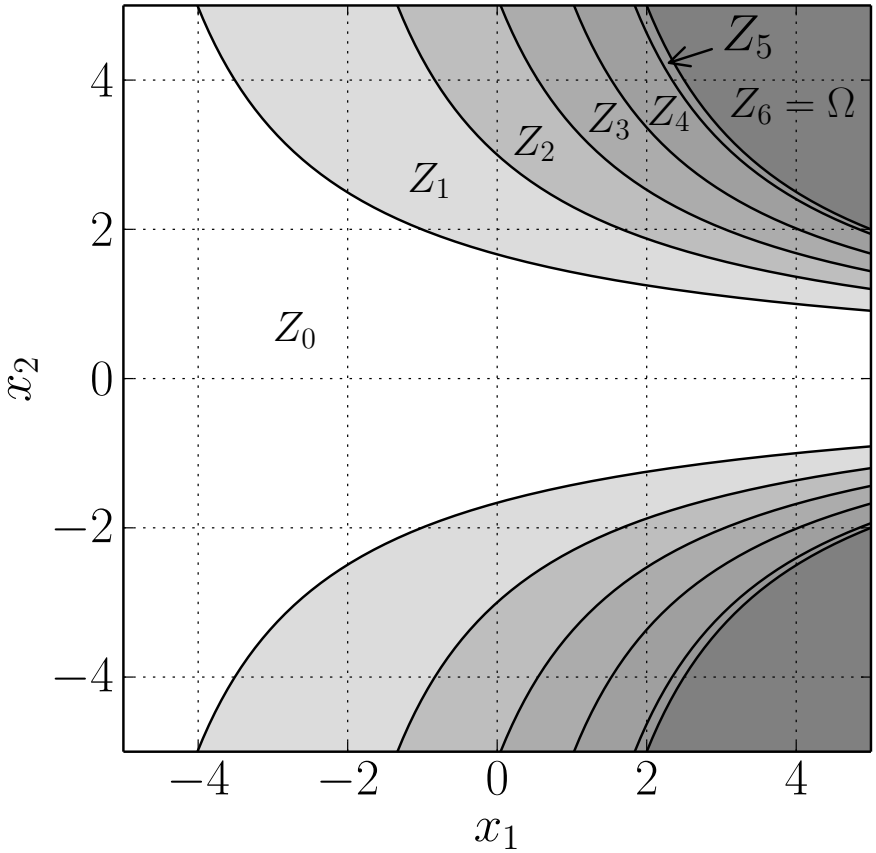
$$P_f = \Pr(\Omega) = \prod_{i=1}^{M} \Pr(Z_i | Z_{i-1})$$

The domains Z_i have to be nested.

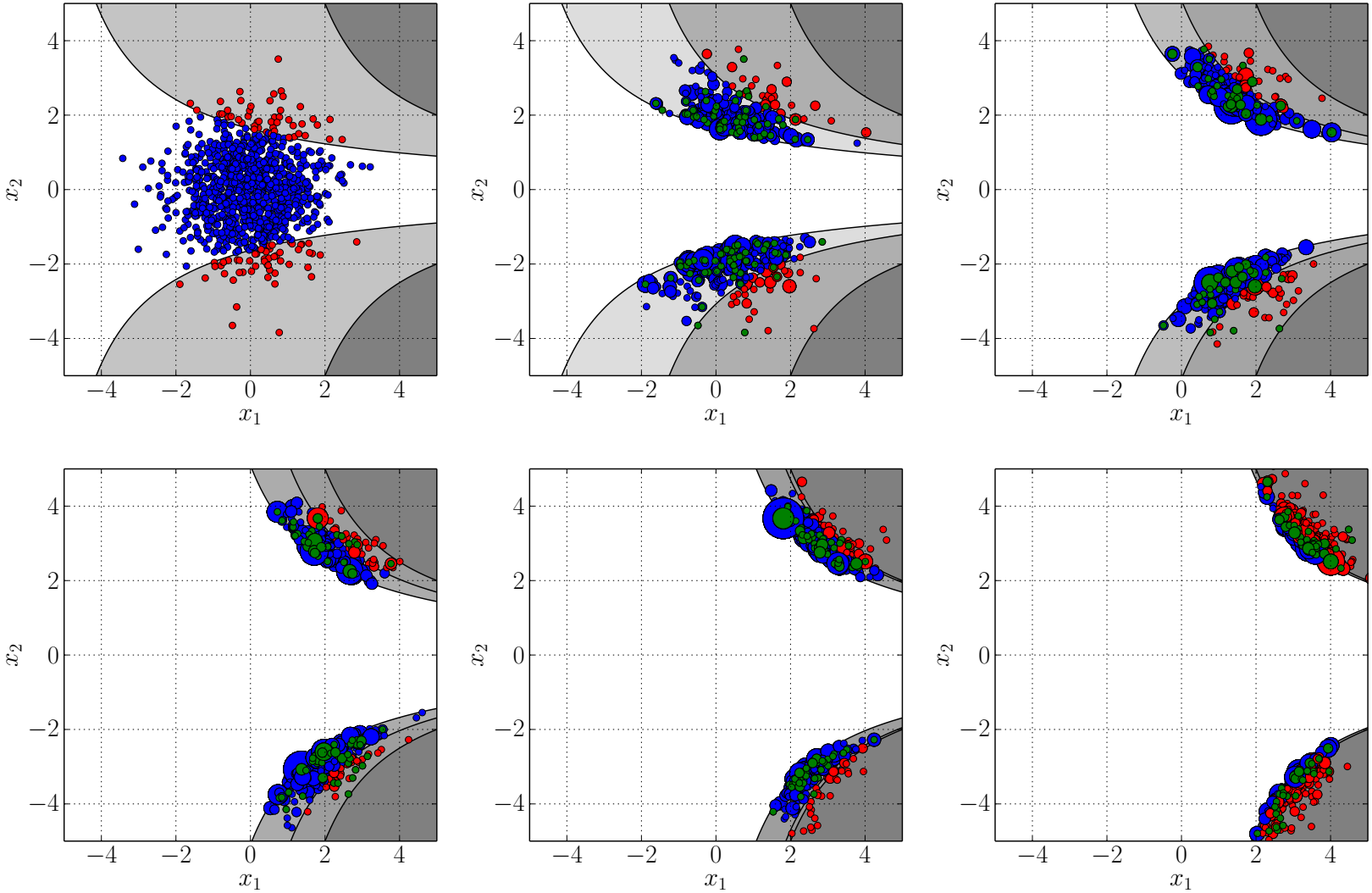
Example:

$$g(\mathbf{x}) = \frac{10}{|x_2|} - x_1$$

x_1 and x_2 are standard Normal



Subset Simulation – general idea (cont'd.)

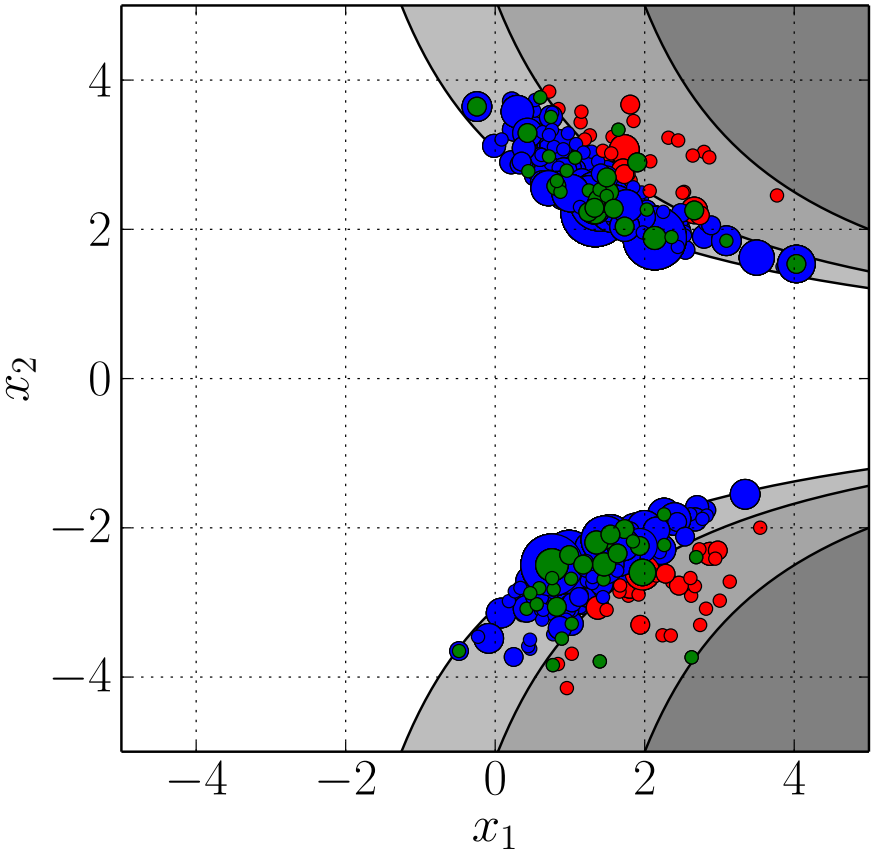


Subset Simulation (SuS) – MCMC sampling

in general, we cannot generate independent samples in the intermediate levels of SuS

efficiency of Subset Simulation depends on the MCMC sampling strategy

- component-wise Metropolis-Hastings
- conditional sampling in standard Normal space



cwMH: component-wise Metropolis-Hastings

[Au and Beck, 2001]

1. Let $\mathbf{w}$ be a vector whose dimension is equal to the dimension of $\mathbf{u}$. For every component $w_n,\ n = 1, \ldots, N$ of $\mathbf{w}$ do:
   1.1 Propose sample ξ_n from proposal distribution $q(\xi_n | u_n^{(i,j,k-1)})$.
   1.2 Accept proposed sample with probability

$$r = \frac{\varphi(\xi_n)}{\varphi(u_n^{(i,j,k-1)})} \cdot \frac{q(u_n^{(i,j,k-1)} | \xi_n)}{q(\xi_n | u_n^{(i,j,k-1)})}$$

   i.e., set $w_n = \xi_n$;
   otherwise set $w_n = u_n^{(i,j,k-1)}$.
2. Evaluate $G(\mathbf{w})$ and check if $\mathbf{w} \in Z_i$.
   2.1 If $\mathbf{w} \in Z_i$: set $\mathbf{u}^{(i,j,k)} = \mathbf{w}$,
   2.2 otherwise set $\mathbf{u}^{(i,j,k)} = \mathbf{u}^{(i,j,k-1)}$.

CS: conditional sampling in standard Normal space

[Papaioannou et al., 2015]

1. Let $\mathbf{w}$ be a vector whose dimension is equal to the dimension of $\mathbf{u}$. For each component $w_n,\ n = 1, \ldots, N$ of $\mathbf{w}$ set w_n as a sample from distribution:

$$q(w_n | u_n^{(i,j,k-1)}) = \varphi\left(\frac{w_n - \rho_q \cdot u_n^{(i,j,k-1)}}{\sqrt{1-\rho_q^2}}\right)$$

   ρ_q: cross-correlation coefficient

2. Evaluate $G(\mathbf{u})$ and check if $\mathbf{w} \in Z_i$.
   2.1 If $\mathbf{w} \in Z_i$: set $\mathbf{u}^{(i,j,k)} = \mathbf{w}$,
   2.2 otherwise set $\mathbf{u}^{(i,j,k)} = \mathbf{u}^{(i,j,k-1)}$.

Conditional sampling with eigendecomposition

Correlation structure of samples at current level can be considered in proposal **w**.

1. Approximate sample-covariance matrix **C** at the beginning of each SuS level.
2. Find eigenvalues λ and eigenvectors **v** of covariance matrix:

$$\mathbf{C}\mathbf{v} = \lambda \mathbf{v}$$

3. Transform current seed **x** as:

$$\mathbf{x}^* = \mathbf{V}\mathbf{x}$$

4. Sample each component w_i^* from a Normal distribution that has
   mean $\rho_q \cdot x_i^*$ and
   variance $1 - {\rho_q}^2$.
5. Get current candidate state:

$$\mathbf{w} = \mathbf{V}^{\mathrm{T}}\mathbf{w}^*$$

6. Check if $\mathbf{w} \in Z_i$

variance of the MCMC proposal distribution

variance too large: many proposed samples are rejected $\rightarrow$ many repeated samples

variance too small: Markov chain produces samples that are close together

$\Rightarrow$ in both cases the generated samples are highly dependent

optimal variance:
variance is optimal if the "velocity" of the chain is maximized (difficult to achieve)

near-optimal variance:
can be achieved by controlling the MCMC acceptance rate
$\rightarrow$ i.e., maintain target acceptance rate
acceptance rate too large $\rightarrow$ increase variance of proposal
acceptance rate too small $\rightarrow$ decrease variance of proposal

adaptive optimization of the spread of the proposal

[Papaioannou et al., 2015]

After every 10th Markov chain, perform the following algorithm:

1. Compute p_{acr} as the number of accepted samples divided by the total number of proposed samples (counted since the last adaptive step)
2. Compute coefficient
$$c_q = \frac{p_{\mathrm{acr}} - t_{\mathrm{acr}}}{\sqrt{N_{\mathrm{adpt}}}}$$
t_{acr}: target-acceptance rate; e.g., 44% [Papaioannou et al., 2015]
N_{adpt}: current number of adaptive steps in the current level of Subset Simulation
3. Modify spread of proposal:
$$\sigma_{\mathrm{new}} = \sigma_{\mathrm{prev}} \cdot \exp\left(c_q\right)$$

Remark

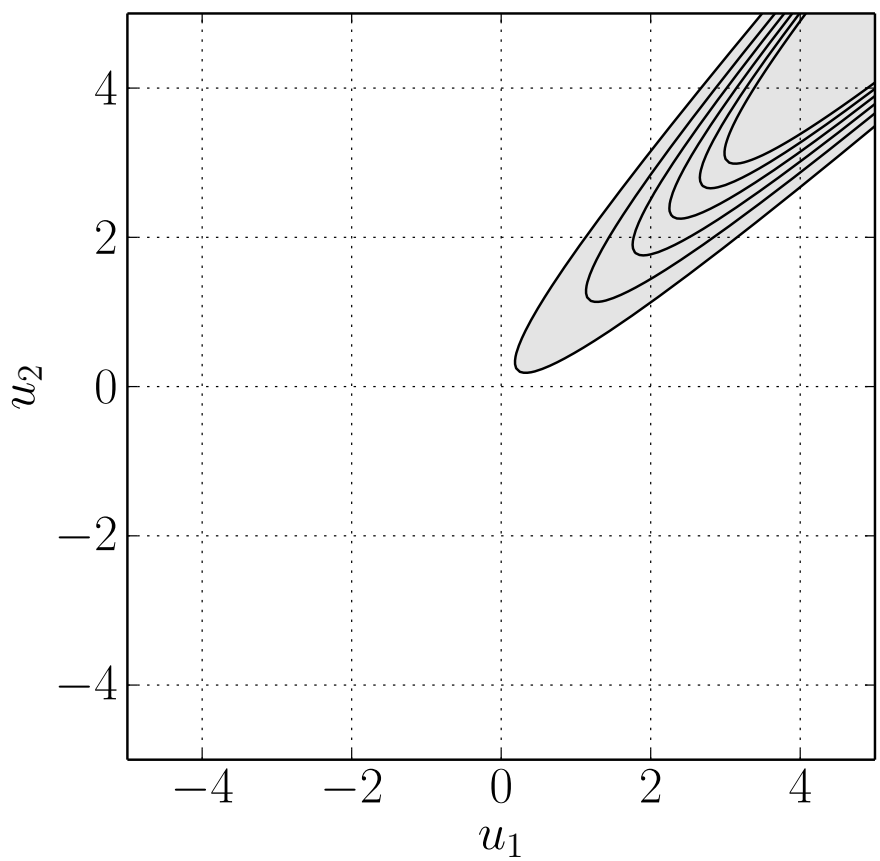
The seeds of the Markov chains should be shuffled randomly in each level of Subset Simulation.

Numerical Investigations

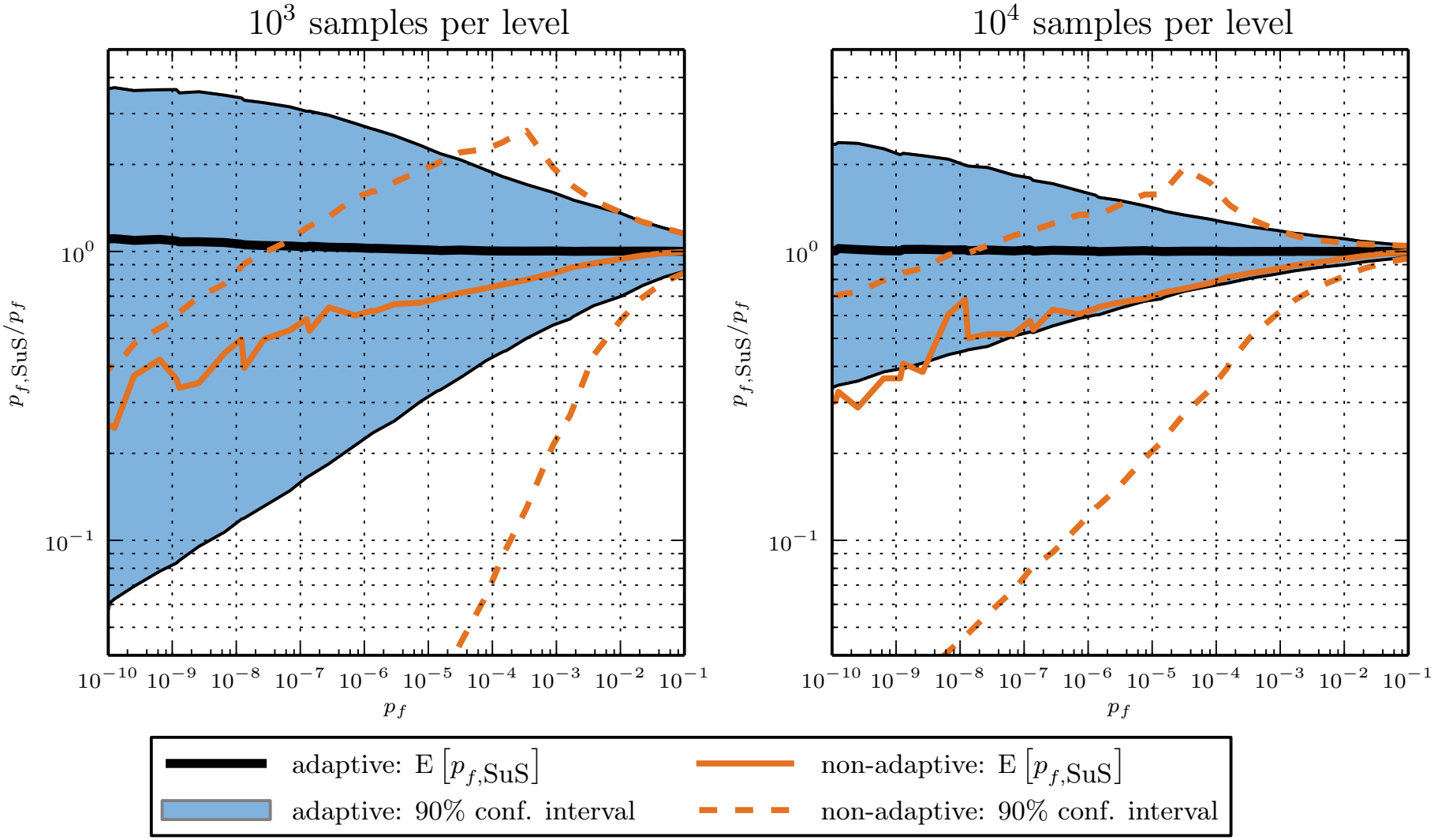
Example problem 1: Gaussian with quadratic term

$$g(\mathbf{x}) = \alpha - \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i + \frac{\kappa}{4}(x_1 - x_2)^2$$

- α distance of design-point to origin
- κ principal curvature at the design point ($\kappa = 10$)
- $\mathbf{x}$ is a N-dimensional vector of independent standard Normal random variables ($N = 10$)



Performance of adapting proposal variance



Example problem 2: hypersphere

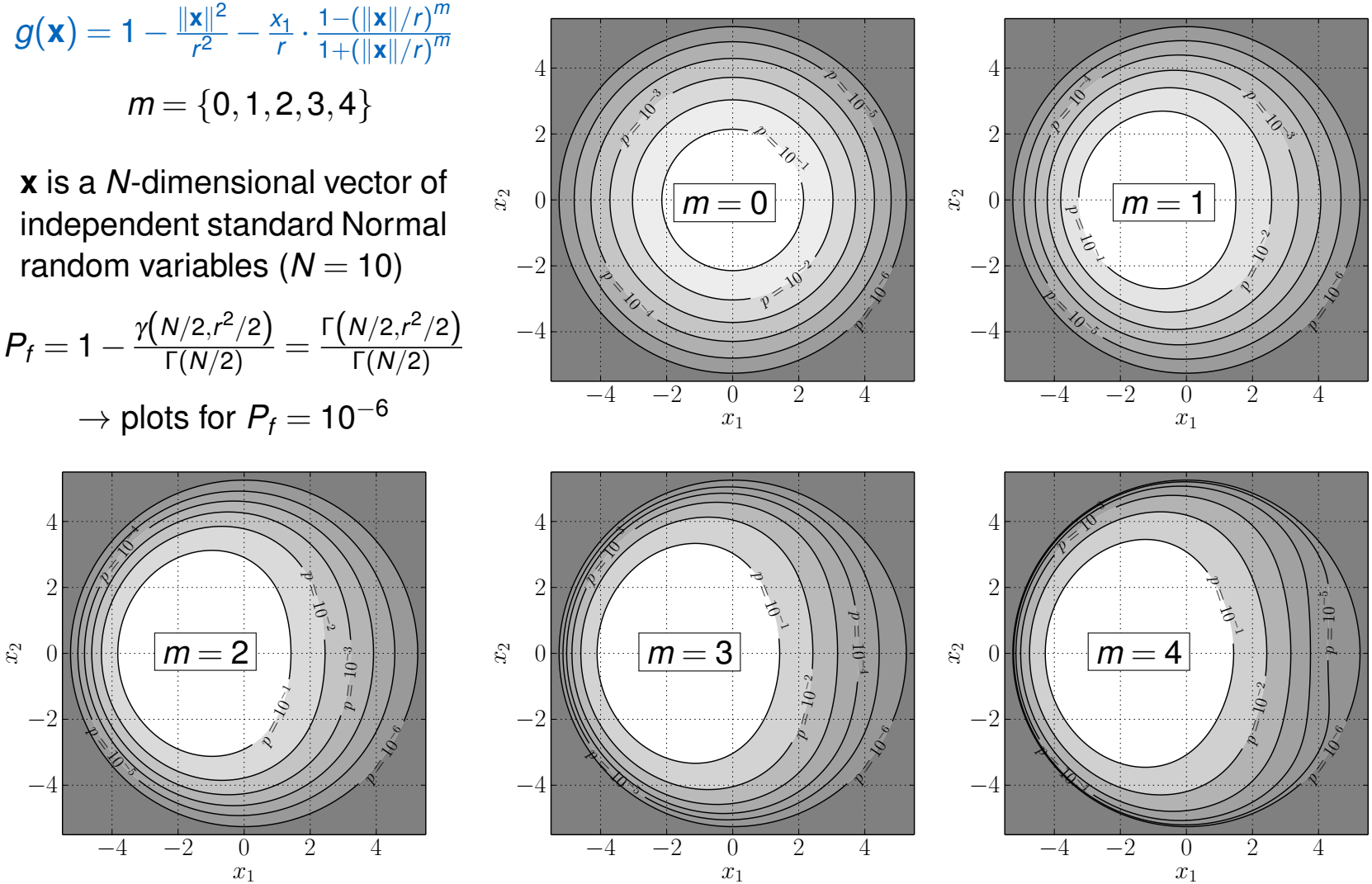
$$g(\mathbf{x}) = 1 - \frac{\|\mathbf{x}\|^2}{r^2} - \frac{x_1}{r} \cdot \frac{1-(\|\mathbf{x}\|/r)^m}{1+(\|\mathbf{x}\|/r)^m}$$

$$m = \{0, 1, 2, 3, 4\}$$

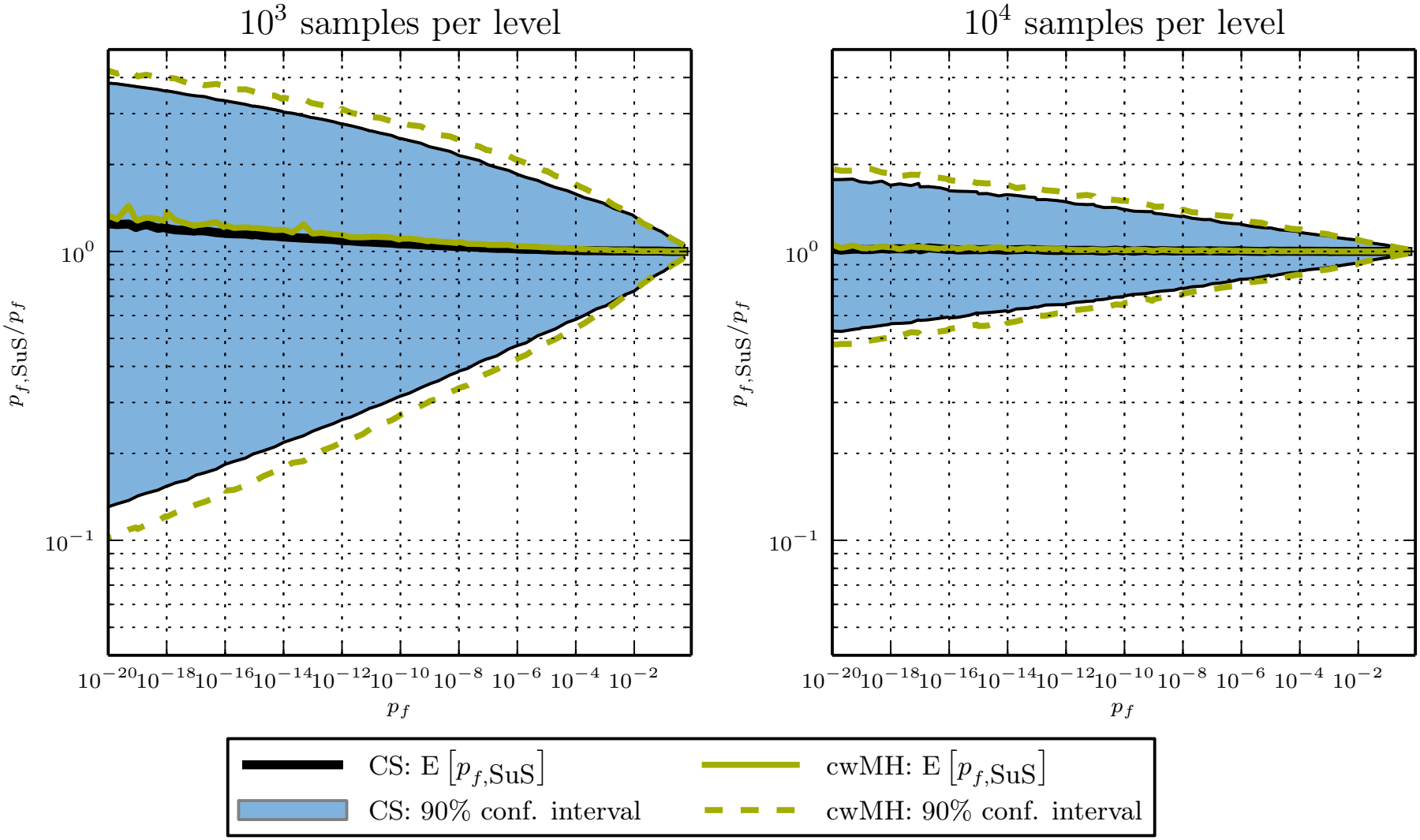
x is a N-dimensional vector of independent standard Normal random variables ($N = 10$)

$$P_f = 1 - \frac{\gamma(N/2, r^2/2)}{\Gamma(N/2)} = \frac{\Gamma(N/2, r^2/2)}{\Gamma(N/2)}$$

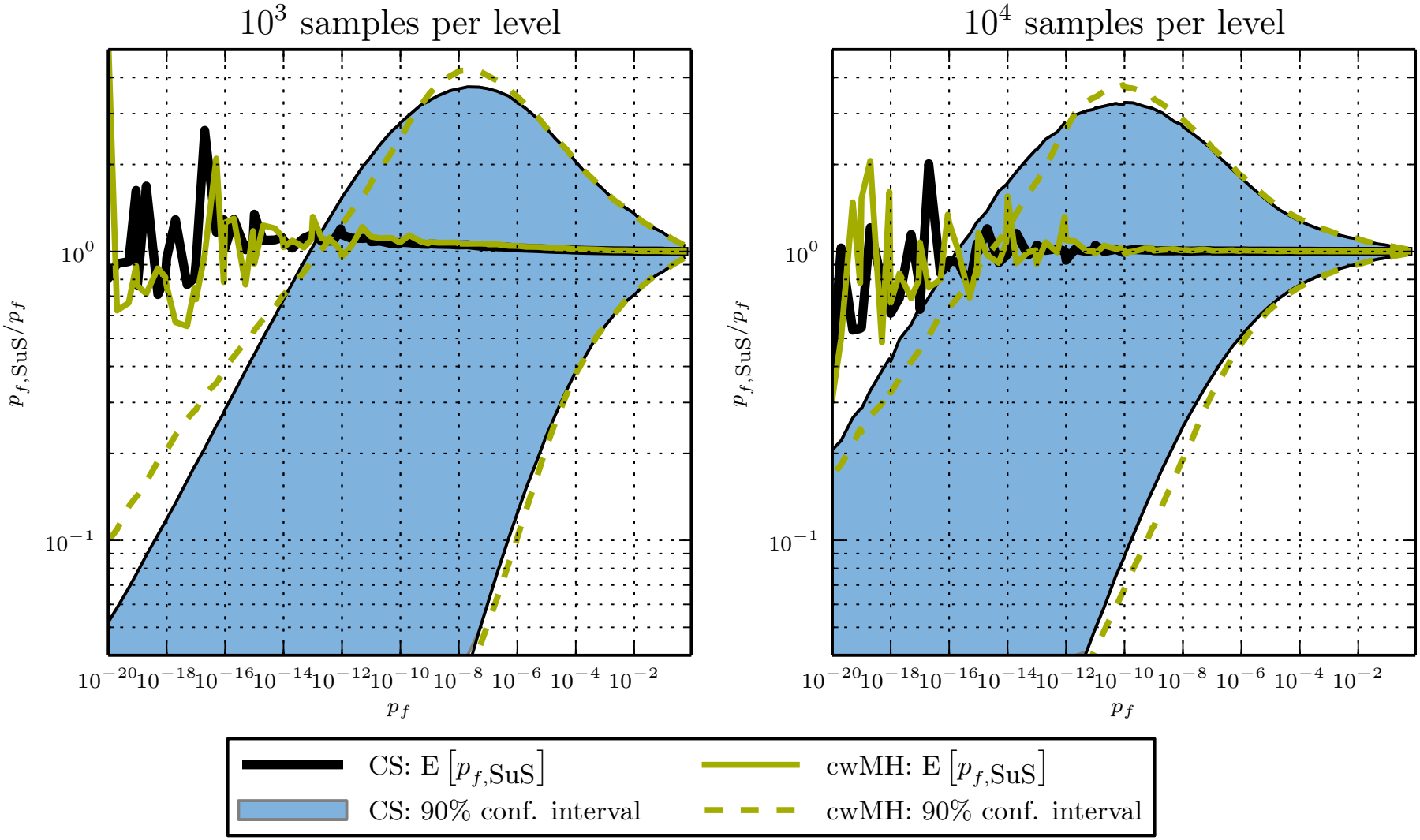
$\rightarrow$ plots for $P_f = 10^{-6}$



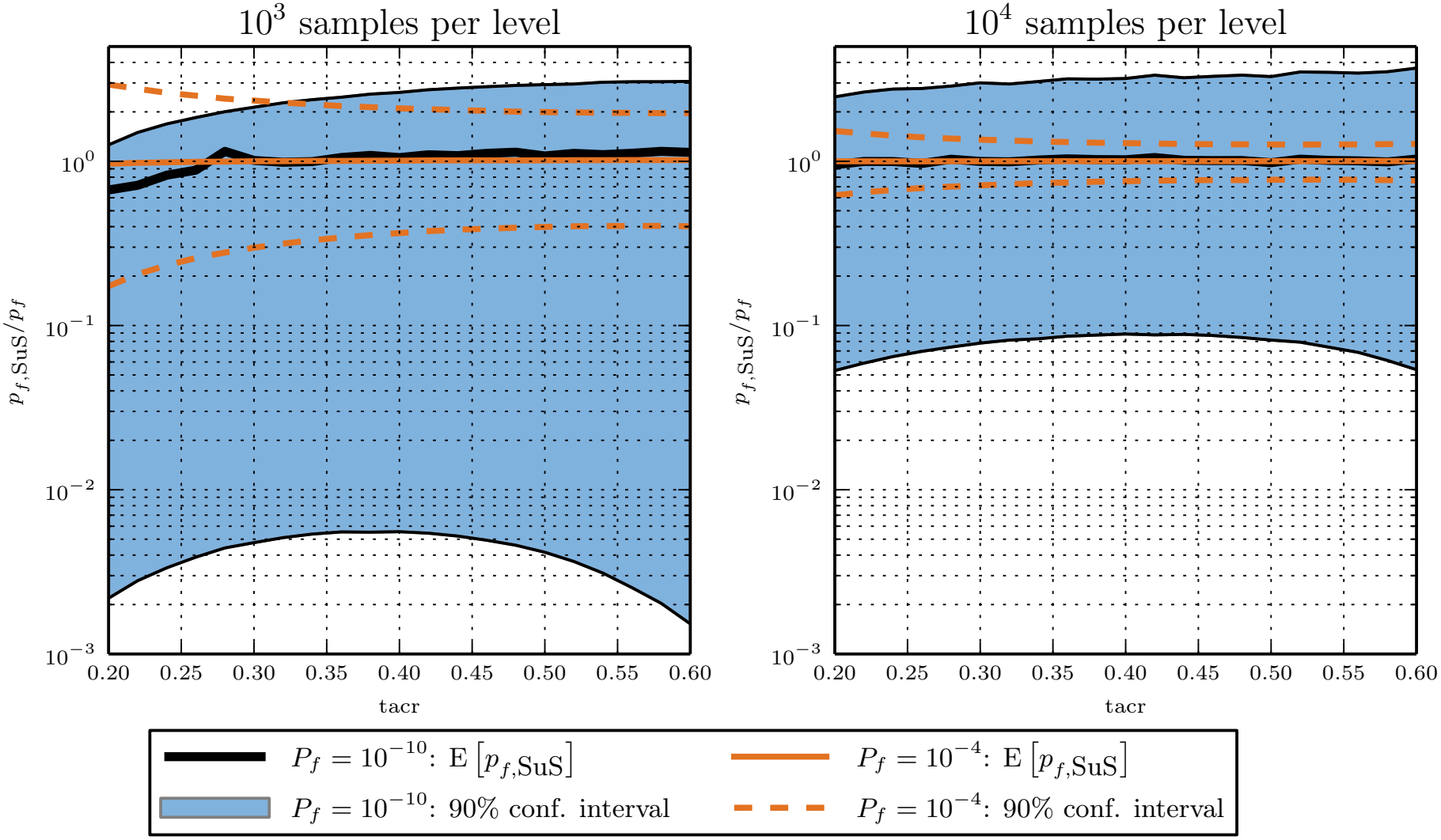
MCMC algorithms (for $m = 0$)



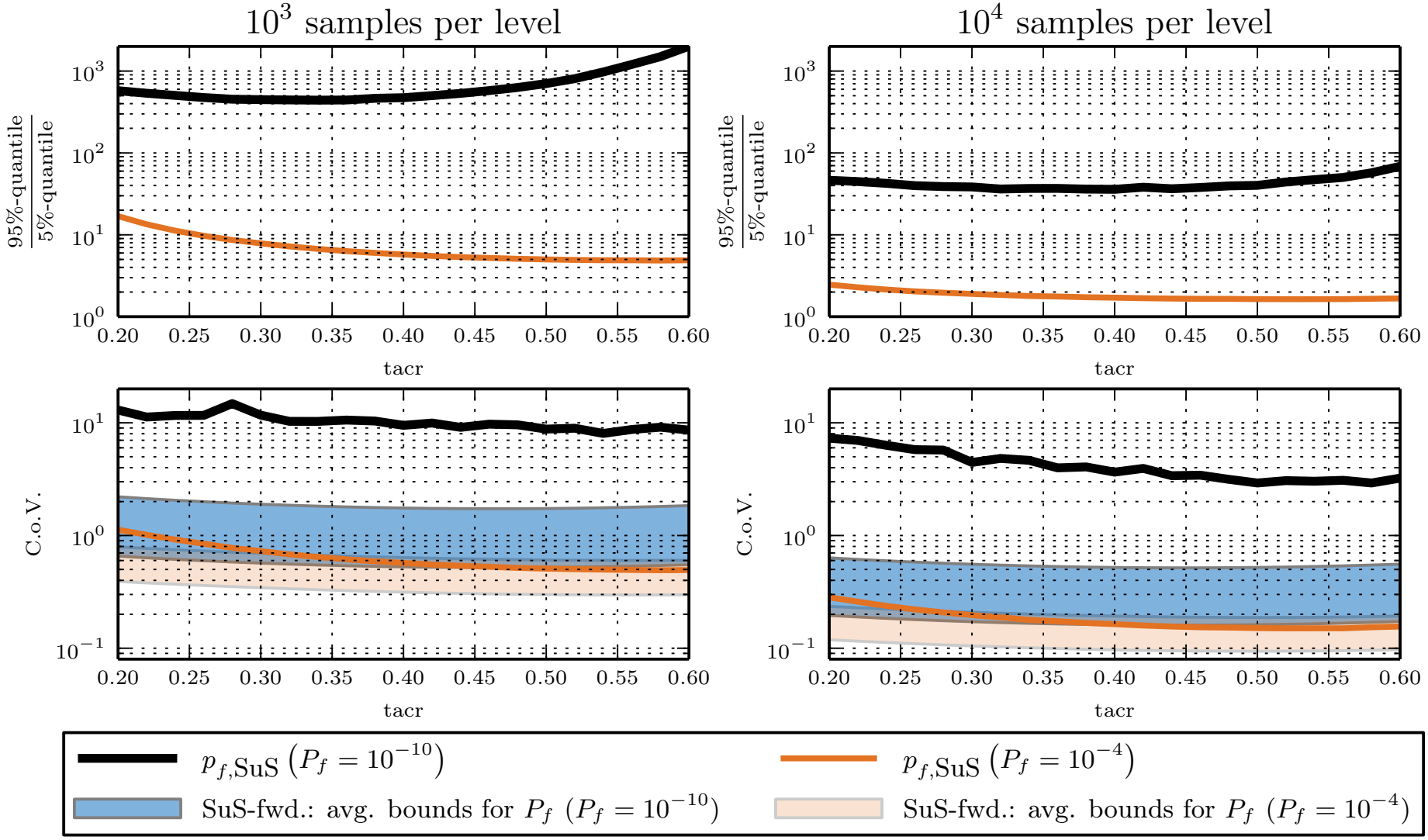
MCMC algorithms (for $m = 4$)



MCMC – target acceptance rate (for $m = 4$)



MCMC – target acceptance rate (for $m = 4$) (cont'd.)



Summary & Conclusion

Conditional sampling in standard Normal space is an interesting alternative to cwMH

By learning the spread of the MCMC proposal adaptively, the performance of SuS can be improved considerably.

Performance of SuS depends on the efficiency of the employed MCMC algorithm.

Dependency of seeds of different MCMC chains (and between different levels) decreases performance of SuS.

Standard estimates for C.o.V. of estimated p_f only consider correlation within MCMC chains.
$\rightarrow$ They should be used with care.

Area for future research: Tools to reliably quantify the uncertainty in the estimated failure probability are still missing.

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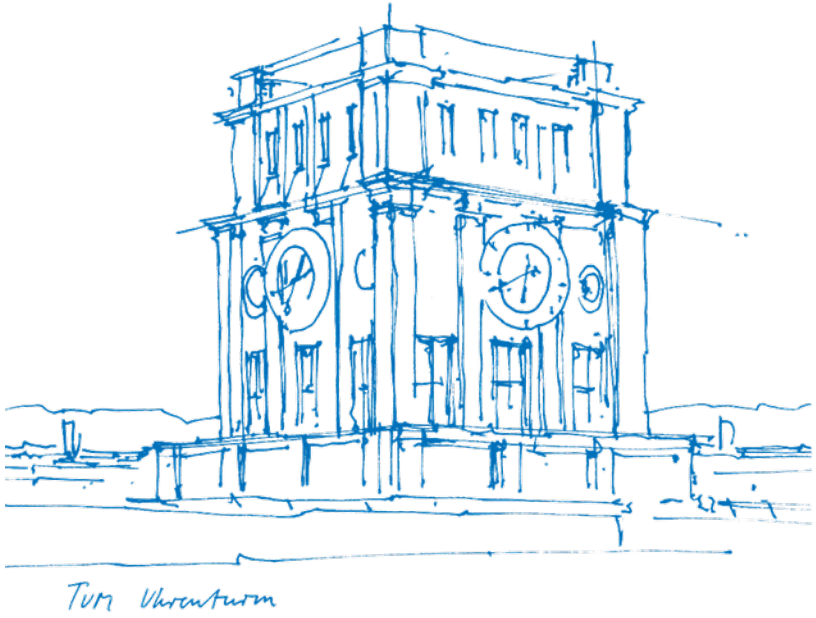
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