An efficient MCMC algorithm for Subset Simulation

Seminars in Engineering Risk Analysis & Probabilistic Modelling

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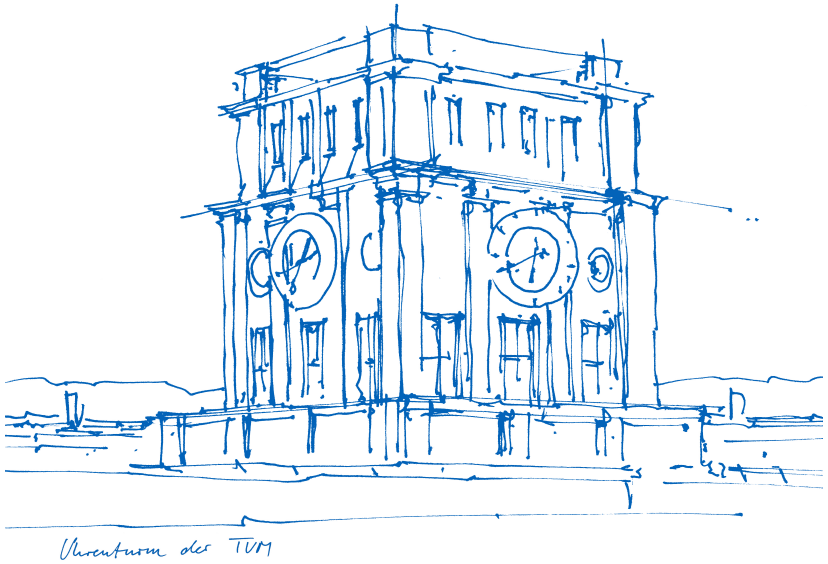


Probability Puzzle

Who is this?

Probability Puzzle – Answer

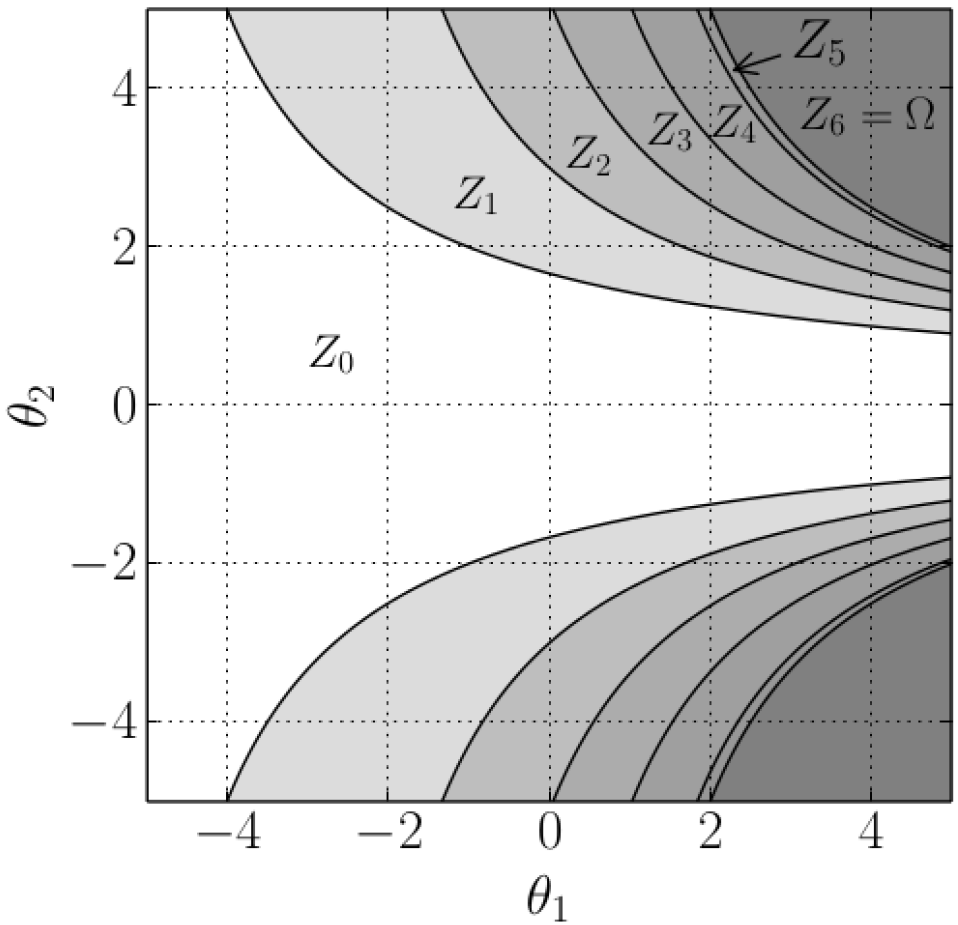
Typically associated with **Thomas Bayes** (1701?-1761)

However, we do not know for sure!!!

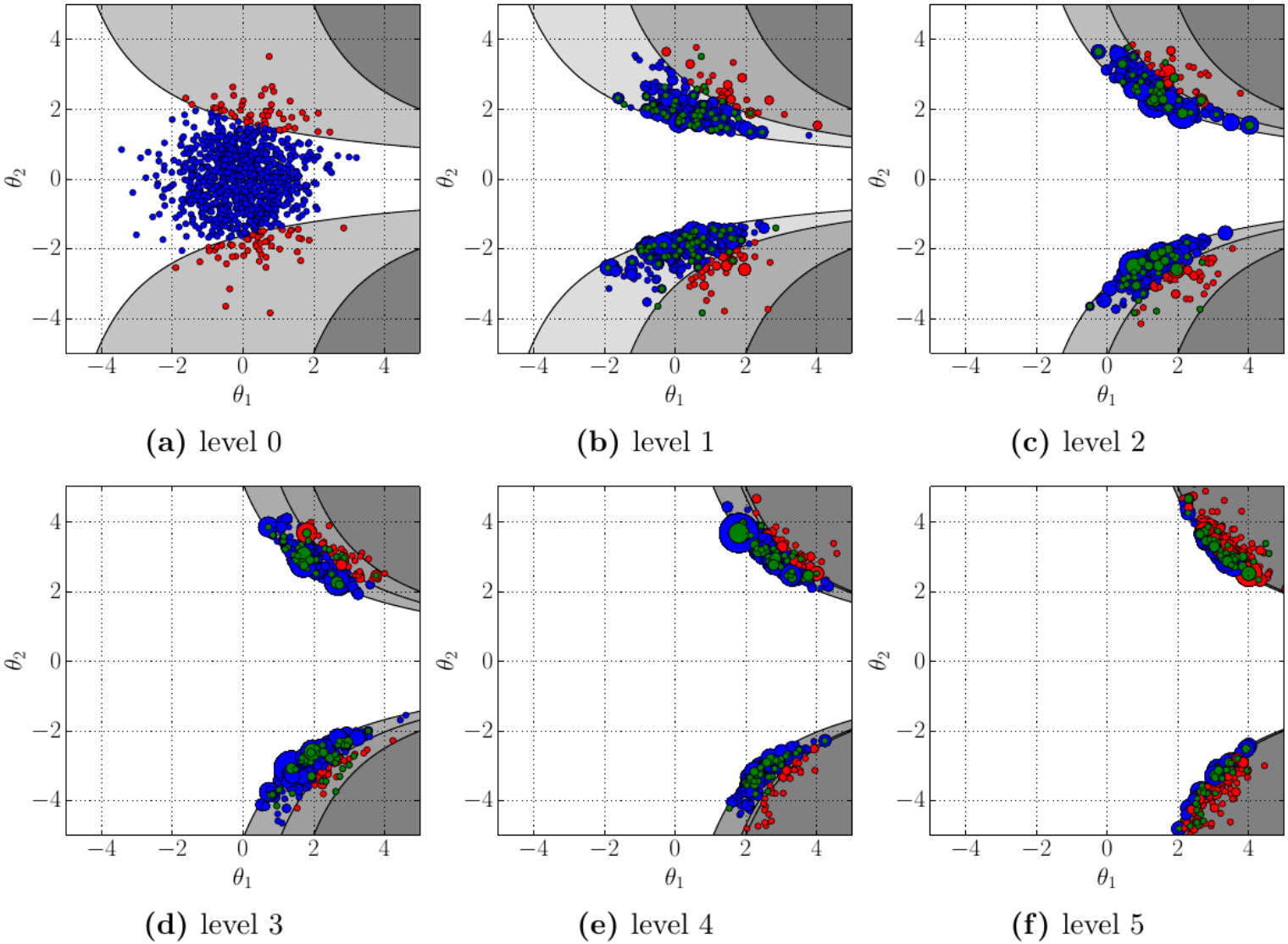
David R. Bellhouse:

"The first thing to note in this picture is the apparent absence of a wig, or if a wig is present, it is definitely the wrong style for the period. [...] The second thing to note is that Bayes appears to be wearing a clerical gown like his father or a larger frock coat with a high collar [...] the gown is not in style for Bayes's generation and the frock coat with a large collar is definitely anachronistic. [...] For reference, I have used C. Willett Cunnington and P. Cunnington, Handbook of English Costume in the Eighteenth Century, pub. Faber & Faber, London, 1964."

Subset Simulation (SuS) – idea



Subset Simulation (SuS) – illustration



(a) level 0 **(b)** level 1 **(c)** level 2

(d) level 3 **(e)** level 4 **(f)** level 5

MCMC in SuS: conditional sampling

[Papaioannou et al., 2015]

1. Let $\mathbf{w}$ be a vector whose dimension is equal to the dimension of $\mathbf{u}$. For each component $w_n,\ n = 1, \ldots, N$ of $\mathbf{w}$ set w_n as a sample from distribution:

$$q(w_n | u_n^{(i,j,k-1)}) = \varphi\left(\frac{w_n - \rho_q \cdot u_n^{(i,j,k-1)}}{\sqrt{1-\rho_q^2}}\right) \tag{3.38}$$

2. Evaluate $g^*(\mathbf{u})$ and check if $\mathbf{w} \in Z_i$.

   (a) If $\mathbf{w} \in Z_i$: set $\mathbf{u}^{(i,j,k)} = \mathbf{w}$,

   (b) otherwise set $\mathbf{u}^{(i,j,k)} = \mathbf{u}^{(i,j,k-1)}$.

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simple MCMC strategy

particularly efficient in high dimensions

MCMC in SuS: spread of proposal distribution

What is the optimal spread?

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optimal spread: uncertainty in estimated p_f is minimized

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How do we measure uncertainty?

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What is ~~the~~ an optimal spread?

optimal spread: uncertainty in estimated p_f is minimized

How do we measure uncertainty?

MCMC in SuS – efficiency measures

$$\mathrm{eff}_{\mathrm{MH},1} = \frac{\mathrm{ESJD}}{\mathrm{ESJD}_{\mathrm{opt}}}$$ expected squared jumping distance

$$\mathrm{eff}_{\mathrm{MH},2} = \mathrm{eff}_{\gamma, I_{G_1}(\boldsymbol{\theta})}$$ indicator function (10%): eff. number of samples

$$\mathrm{eff}_{\mathrm{MH},3} = \mathrm{eff}_{\gamma, I_{G_2}(\boldsymbol{\theta})}$$ indicator function (50%): eff. number of samples

$$\mathrm{eff}_{\mathrm{MH},4} = \mathrm{eff}_{\frac{1}{M}\sum_{i=1}^{M}\theta_i}$$ variance of sample mean: eff. number of samples

$$\mathrm{eff}_{\mathrm{MH},5} = \mathrm{eff}_{\theta_1}$$ variance of sample mean: eff. number of samples

MCMC in SuS – Example 1

one-dimensional problem

truncated standard Normal target distribution

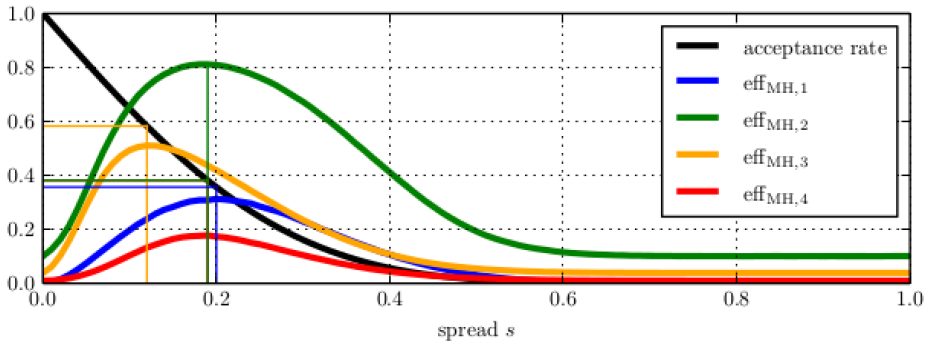
on support $[9,\infty)$

rejection sampling acceptance rate: 10^{-19}

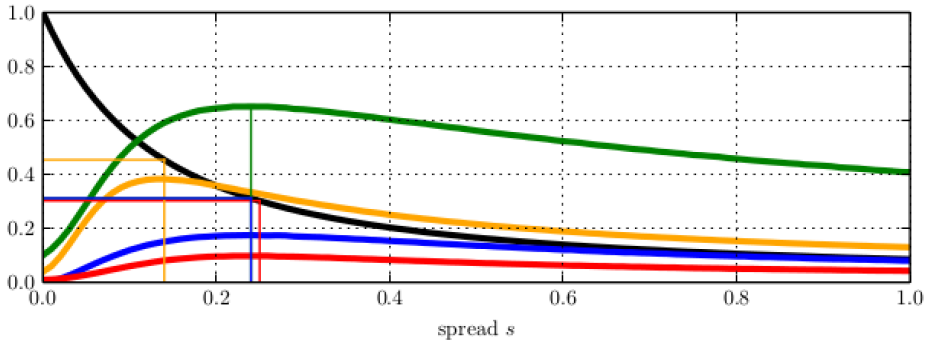
<u>We investigate:</u> optimal acceptance rates (and spreads) for different error measures.

MCMC in SuS – Example 1

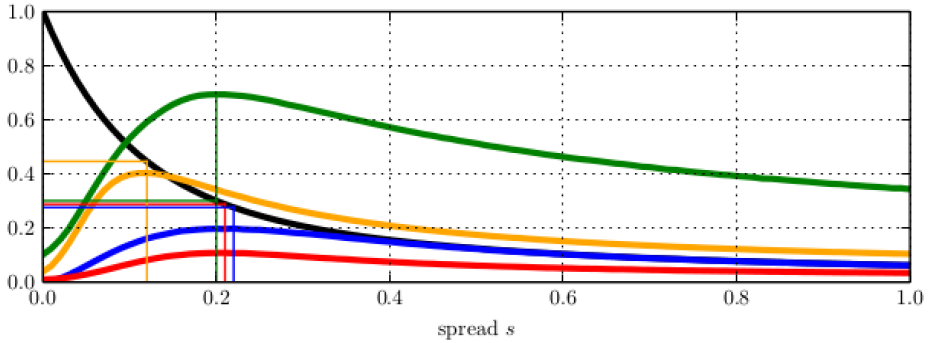
conditional sampling



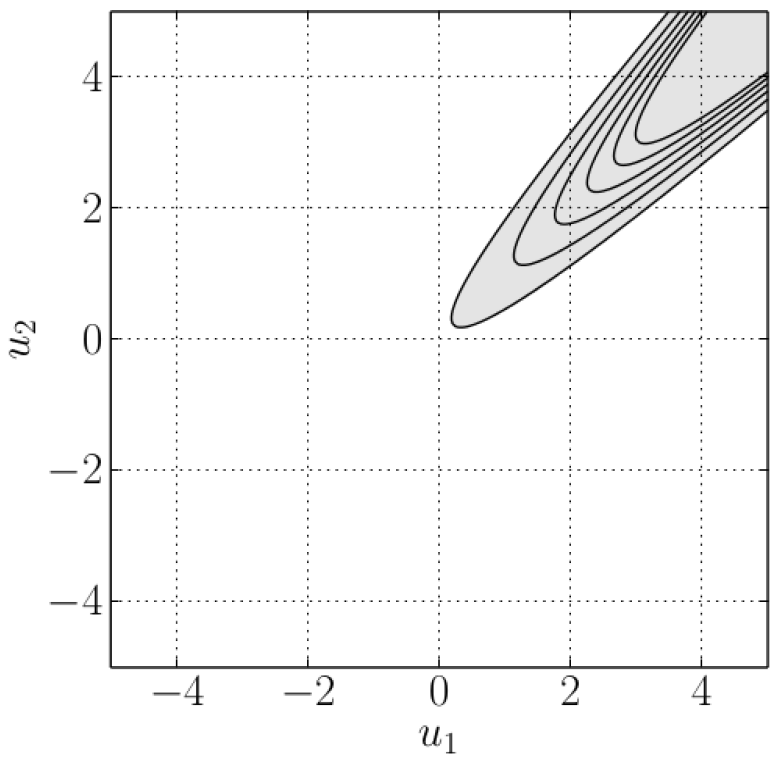
RWM (Normal)



RWM (uniform)



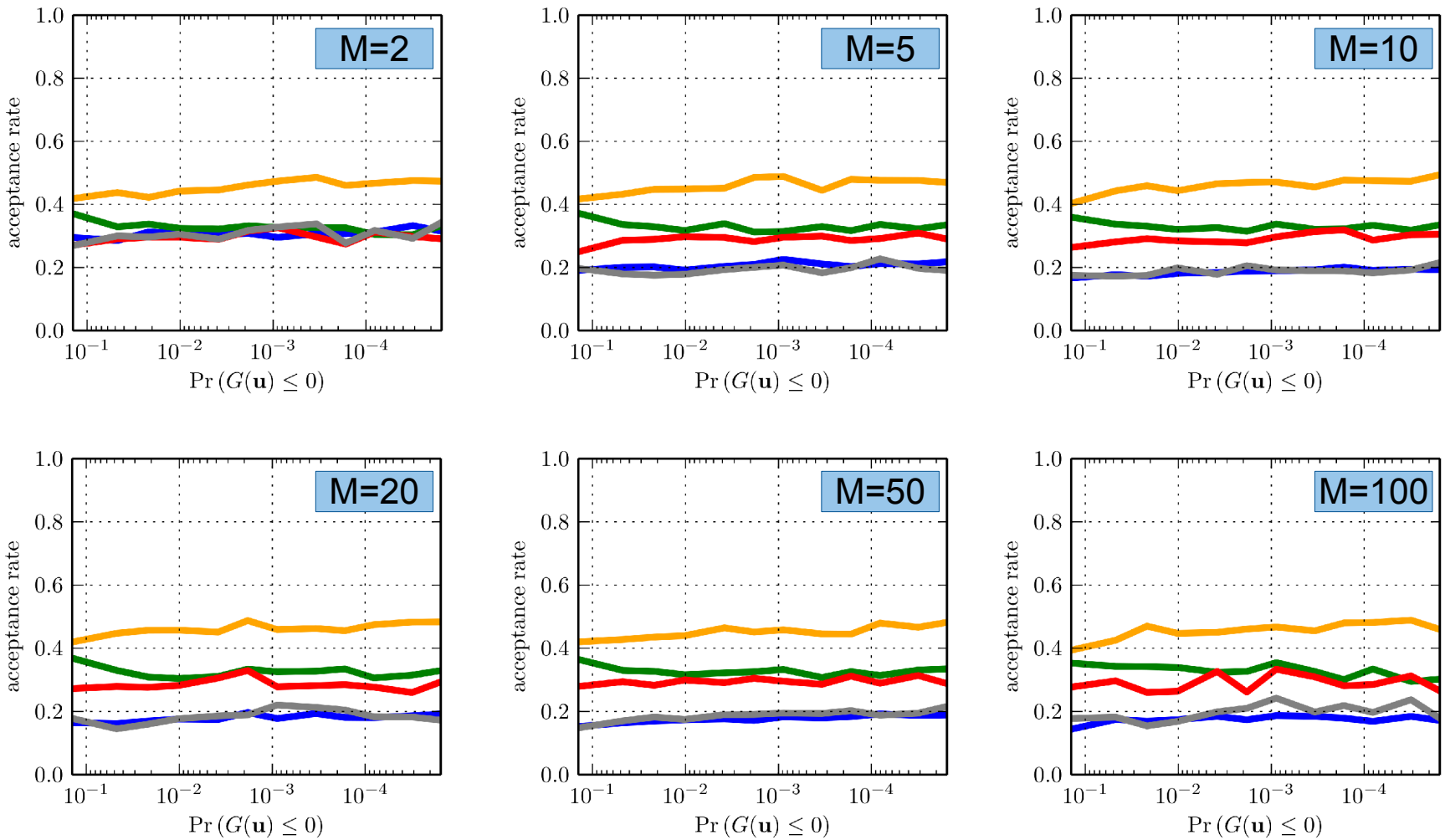
MCMC in SuS – Example 2



$M = 10$

$$G(\mathbf{u}) = t - \frac{1}{M}\sum_{i=1}^{M} u_i + \frac{10}{4}\left(u_1 - u_2\right)^2$$

MCMC in SuS – Example 2

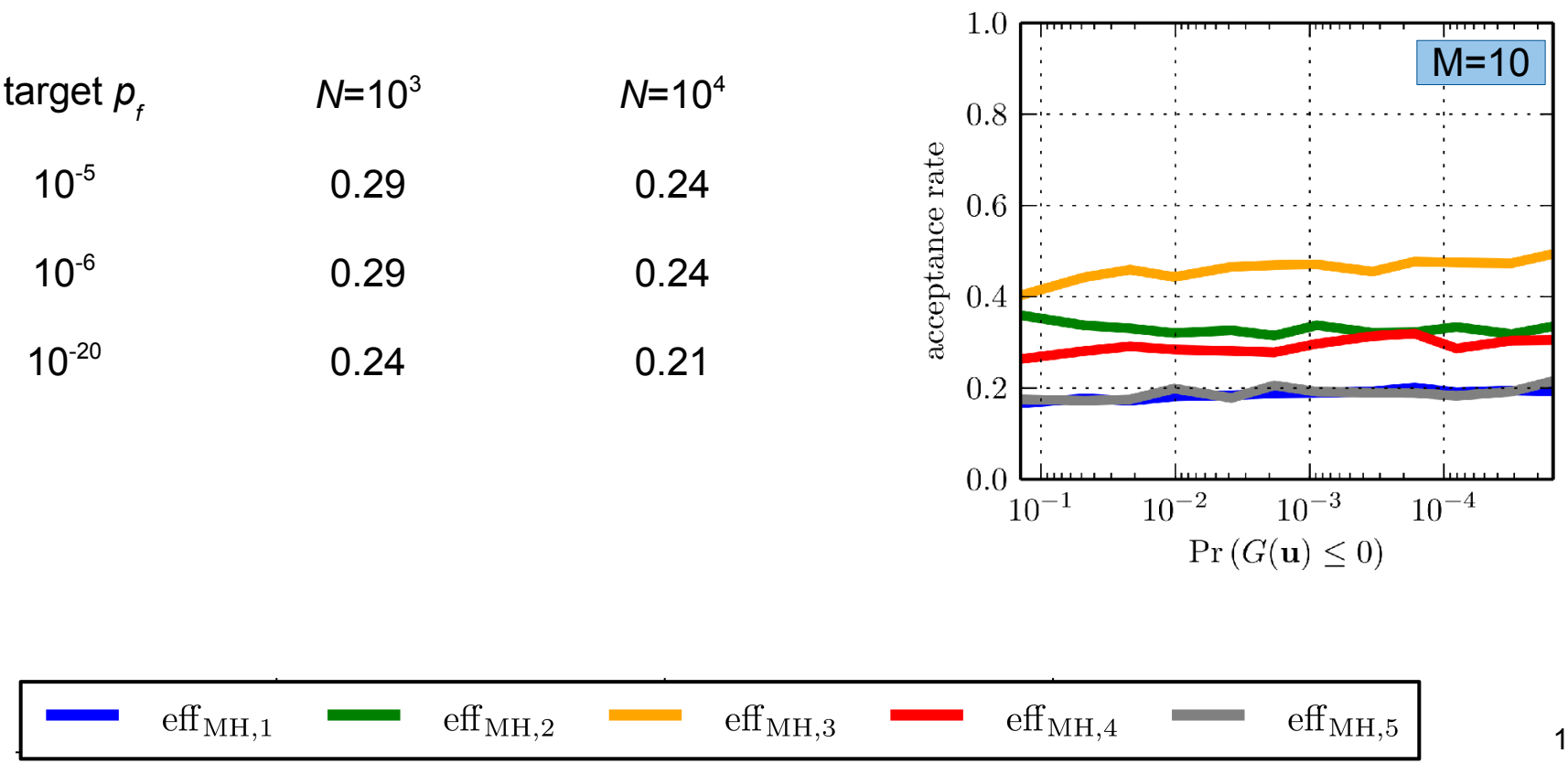


$\mathrm{eff}_{\mathrm{MH},1}$ $\mathrm{eff}_{\mathrm{MH},2}$ $\mathrm{eff}_{\mathrm{MH},3}$ $\mathrm{eff}_{\mathrm{MH},4}$ $\mathrm{eff}_{\mathrm{MH},5}$

MCMC in SuS – Example 2 (contd.)

optimal acceptance rate within SuS for Example 2:

target p_f	N=10^3	N=10^4
10^{-5}	0.29	0.24
10^{-6}	0.29	0.24
10^{-20}	0.24	0.21



Summary

An acceptance rate of 0.44 is a robuts choide.

However, finding an optimal acceptance rate can further improve the efficiency of SuS.

But … the optimal acceptance rate depends on the investigated problem.

Moreover, it is not clear how to optimally measure MCMC efficiency in SuS.

<u>Future work:</u> Learn optimal acceptance rate through importance sampling in SuS.

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