Assessment of methods for the numerical solution of the Fredholm integral eigenvalue problem in the Karhunen-Loève expansion

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Workshop
Structural Reliability, Risk Assessment and Decision-Making: Past, Present, Future

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Probability Puzzle

Four balls are placed in a bag. One is white, one is blue and the other two are red. The bag is shaken and someone draws two balls from it. He looks at the two balls and announces that at least one of them is red.
What are the chances that the other ball he has drawn out is also red?

Probability Puzzle - Solution

R**R**
RW
RB
RW
RB
(BW)

Result: $\mathbf{20}\%$

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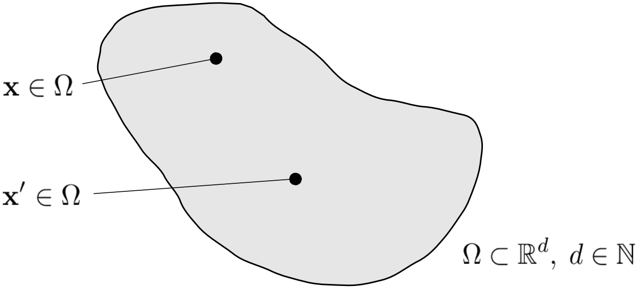
Notation



- random field (RF): $H(\mathbf{x})$
- *Gaussian* random field - completely described by:
  - mean function $\mu(\mathbf{x})$
  - covariance function $\mathrm{Cov}(\mathbf{x}, \mathbf{x}') = \sigma(\mathbf{x}) \cdot \sigma(\mathbf{x}') \cdot \rho(\mathbf{x}, \mathbf{x}')$
    - $\sigma(\mathbf{x})$: standard deviation function
    - $\rho(\mathbf{x}, \mathbf{x}')$: correlation coefficient function

Random field discretization

Number of random variables (RVs) in a random field

- theoretically: *infinite* number of RVs (∞)
  - for each $\mathbf{x} \in \Omega$, $H(\mathbf{x})$ represents a RV
- discretized RF: *finite* number of RVs (M)

$$H(\mathbf{x}) \xrightarrow{\text{discretization}} \hat{H}(\mathbf{x}) \tag{1}$$

Categories of RF-discretization methods

- point discretization methods
- averaging discretization methods
- **series expansion methods**
  - Karhunen-Loève (KL) expansion
  - EOLE method

Karhunen-Loève expansion

KL-expansion

$$H(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, \varphi_i(\mathbf{x})\, \xi_i \tag{2}$$

- λ_i: eigenvalues of the covariance kernel
- φ_i: eigenfunctions of the covariance kernel
  - orthonormal: $\int_\Omega \varphi_i(\mathbf{x})\varphi_j(\mathbf{x})\,\mathrm{d}\mathbf{x} = \delta_{ij}$
- ξ_i: uncorrelated standard normal RVs
  - orthonormal: $\mathrm{E}[\xi_i \xi_j] = \delta_{ij}$

Integral eigenvalue problem

$$\int\limits_{\mathbf{x}'\in\Omega} \varphi_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}' = \lambda_i \varphi_i(\mathbf{x}) \tag{3}$$

Truncated KL-expansion

KL-expansion *(exact representation)*

$$H(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, \varphi_i(\mathbf{x})\, \xi_i \quad (4)$$

Truncated KL-expansion *(approximation)*

$$\tilde{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\lambda_i}\, \varphi_i(\mathbf{x})\, \xi_i \quad (5)$$

- λ_i: M **largest** eigenvalues (in descending order)

. . . optimal with respect to:

$$\int_\Omega \mathrm{E}\left[\left(H(\mathbf{x}) - \tilde{H}(\mathbf{x})\right)^2\right] \mathrm{d}\mathbf{x} \quad (6)$$

Fredholm Integral Eigenvalue Problem

Fredholm integral eigenvalue problem (KL-expansion)

$$\int_{\mathbf{x}' \in \Omega} \varphi_i(\mathbf{x}') \mathrm{Cov}(\mathbf{x}, \mathbf{x}') \, \mathrm{d}\mathbf{x}' = \lambda_i \varphi_i(\mathbf{x}) \tag{7}$$

Numerical approximation

- . . . of $\hat{\lambda}_i$ and $\hat{\varphi}_i(\mathbf{x})$

Approximate KL-expansion

$$\tilde{H}(\mathbf{x}) \approx \hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\hat{\lambda}_i} \, \hat{\varphi}_i(\mathbf{x}) \, \xi_i \tag{8}$$

Numerical Methods to Solve the Integral EVP

Numerical methods

- **Nyström method**
- Projection methods
  - Collocation method
  - **Galerkin methods**
- Degenerate kernel methods

Nyström method

Fredholm integral eigenvalue problem (KL-expansion)

$$\int_{\mathbf{x}'\in\Omega} \varphi_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}' = \lambda_i\varphi_i(\mathbf{x}) \tag{9}$$

Approximate integral eigenvalue problem

$$\sum_{j=1}^{N} w_j\,\mathrm{Cov}(\mathbf{x},\mathbf{x}_j)\,\hat{\varphi}_i(\mathbf{x}_j) = \hat{\lambda}_i\hat{\varphi}_i(\mathbf{x}) \tag{10}$$

Matrix eigenvalue problem

$$\sum_{j=1}^{N} w_j\,\mathrm{Cov}(\mathbf{x}_k,\mathbf{x}_j)\,\hat{\varphi}_i(\mathbf{x}_j) = \hat{\lambda}_i\hat{\varphi}_i(\mathbf{x}_k), \quad k=1,\ldots,N \tag{11}$$

Nyström method - Solution

Matrix eigenvalue problem

$$\mathbf{C}\mathbf{W}\mathbf{y}_i = \hat{\lambda}_i \mathbf{y}_i \tag{12}$$

$$\mathbf{W}^{\frac{1}{2}}\mathbf{C}\mathbf{W}^{\frac{1}{2}}\mathbf{y}_i^* = \hat{\lambda}_i \mathbf{y}_i^* \tag{13}$$

- $\mathbf{C}_{kj} = \mathrm{Cov}(\mathbf{x}_k, \mathbf{x}_j)$
- $\mathbf{W}_{ij} = \delta_{ij} w_j \qquad \mathbf{W}^{\frac{1}{2}} = \delta_{ij}\sqrt{w_j}$

Approximation of the eigenfunctions

$$\hat{\varphi}_i(\mathbf{x}) = \frac{1}{\hat{\lambda}_i} \sum_{j=1}^{N} \sqrt{w_j}\, \mathrm{Cov}(\mathbf{x}, \mathbf{x}_j)\ (\mathbf{y}_i^*)_j \tag{14}$$

Normalization of the eigenfunctions

$$\|\mathbf{y}_i^*\| = 1 \tag{15}$$

EOLE method [Li & Der Kiureghian, 1993]

Matrix eigenvalue problem

$$\mathbf{C}\mathbf{y}_i^* = \hat{\lambda}_i^* \mathbf{y}_i^* \tag{16}$$

- $\mathbf{C}_{kj} = \mathrm{Cov}(\mathbf{x}_k, \mathbf{x}_j)$

Random field approximation

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \frac{1}{\sqrt{\hat{\lambda}_i^*}} \sum_{j=1}^{N} \mathrm{Cov}(\mathbf{x}, \mathbf{x}_j)\ (\mathbf{y}_i^*)_j \tag{17}$$

Normalization of the eigenvectors

$$\|\mathbf{y}_i^*\| = 1 \tag{18}$$

Equivalence of Nyström and EOLE

Matrix eigenvalue problem

$$\mathbf{CW}\mathbf{y}_i = \hat{\lambda}_i \mathbf{y}_i \;\Leftrightarrow\; \frac{|\Omega|}{N}\mathbf{CI}\mathbf{y}_i = \hat{\lambda}_i \mathbf{y}_i \;\Leftrightarrow\; \mathbf{C}\mathbf{y}_i^* = \hat{\lambda}_i^* \mathbf{y}_i^* \tag{19}$$

Nyström - Random field approximation

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\frac{|\Omega|}{N\hat{\lambda}_i}} \sum_{j=1}^{N} \mathrm{Cov}(\mathbf{x}, \mathbf{x}_j)\; (\mathbf{y}_i^*)_j \tag{20}$$

EOLE - Random field approximation

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \frac{1}{\sqrt{\hat{\lambda}_i^*}} \sum_{j=1}^{N} \mathrm{Cov}(\mathbf{x}, \mathbf{x}_j)\; (\mathbf{y}_i^*)_j \tag{21}$$

Projection methods

Residuum

$$\int_{\mathbf{x}'\in\Omega} \hat{\varphi}_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}' - \hat{\lambda}_i\hat{\varphi}_i(\mathbf{x}) = \varepsilon_r(\mathbf{x}) \tag{22}$$

Collocation method - pointwise minimization

$$\int_{\mathbf{x}\in\Omega} \varepsilon_r(\mathbf{x})\,\delta(\mathbf{x}_k)\,\mathrm{d}\mathbf{x} = 0 \quad \forall\, k = 1,\dots,N \tag{23}$$

Galerkin method

$$\int_\Omega \varepsilon_r(\mathbf{x})N_k(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \quad \forall\, k = 1,\dots,N \tag{24}$$

Matrix eigenvalue problem

Matrix eigenvalue problem

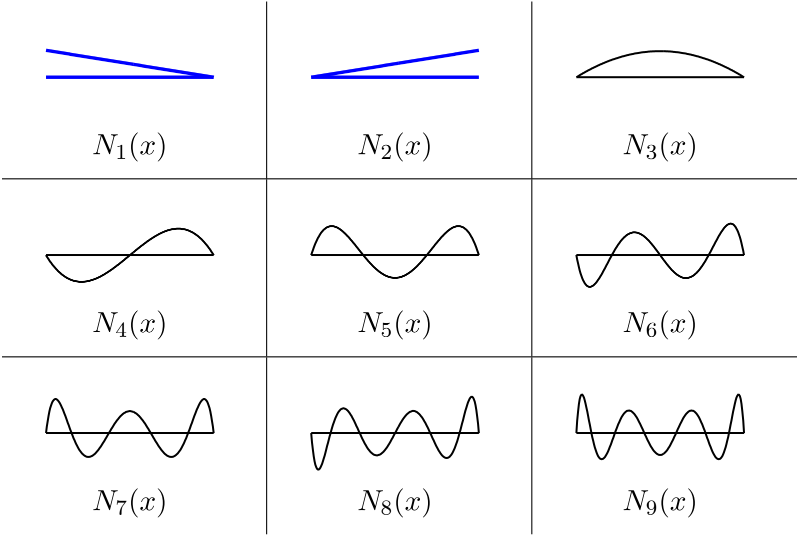
$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \tag{25}$$

where

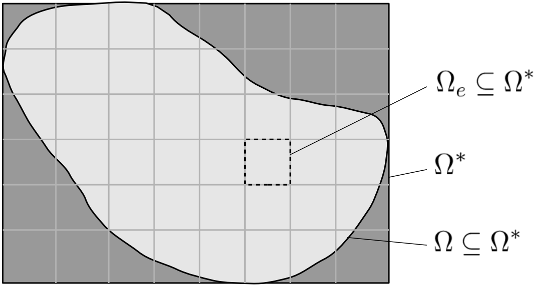
$$B_{kn} = \int_{\mathbf{x}\in\Omega} N_k(\mathbf{x}) \int_{\mathbf{x}'\in\Omega} N_n(\mathbf{x}') \mathrm{Cov}(\mathbf{x},\mathbf{x}') \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \tag{26}$$

$$M_{ij} = \int_{\mathbf{x}\in\Omega} N_i(\mathbf{x}) N_j(\mathbf{x}) \, \mathrm{d}\mathbf{x} \tag{27}$$

Hierarchic Gegenbauer polynomials (1D)



Finite cell - notation



- global shape functions: $N_i \in L^2(\Omega^*)$

$$\alpha(\mathbf{x}) = \begin{cases} 1 & \forall \mathbf{x} \in \Omega \\ 0 & \forall \mathbf{x} \in \Omega^* \setminus \Omega \end{cases} \tag{28}$$

Finite cell approach of the pFEM-KL-expansion

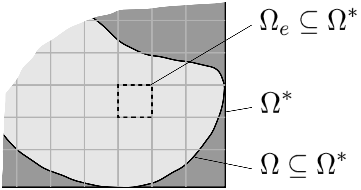
Matrix eigenvalue problem

$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \quad (29)$$

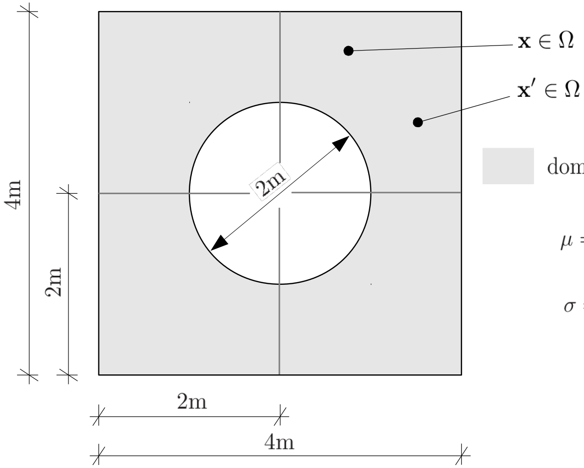
where

$$B_{kn} = \int_{\mathbf{x}\in\Omega^*} \alpha(\mathbf{x}) N_k(\mathbf{x}) \int_{\mathbf{x}'\in\Omega^*} \alpha(\mathbf{x}') N_n(\mathbf{x}') \operatorname{Cov}(\mathbf{x},\mathbf{x}') \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \quad (30)$$

$$M_{ij} = \int_{\mathbf{x}\in\Omega^*} \alpha(\mathbf{x}) N_i(\mathbf{x}) N_j(\mathbf{x}) \, \mathrm{d}\mathbf{x} \quad (31)$$



Example of a plate with a hole

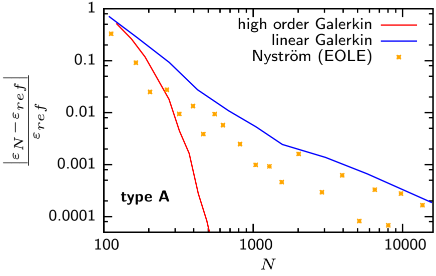


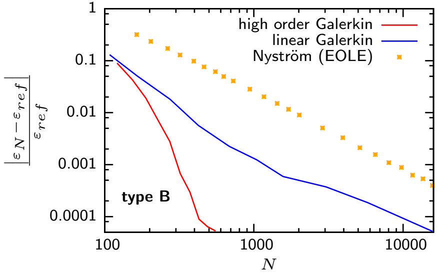
domain Ω of the random field with

$$\mu = 30 \cdot 10^3 \frac{\text{MN}}{\text{m}^2}$$

$$\sigma = 6 \cdot 10^3 \frac{\text{MN}}{\text{m}^2}$$

$M = 100$: relative error



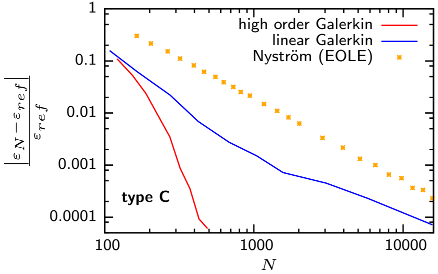


A: $\rho(\mathbf{x},\mathbf{x}') = \exp\left(-\left(\frac{|\mathbf{x}-\mathbf{x}'|}{0.3325}\right)^2\right)$

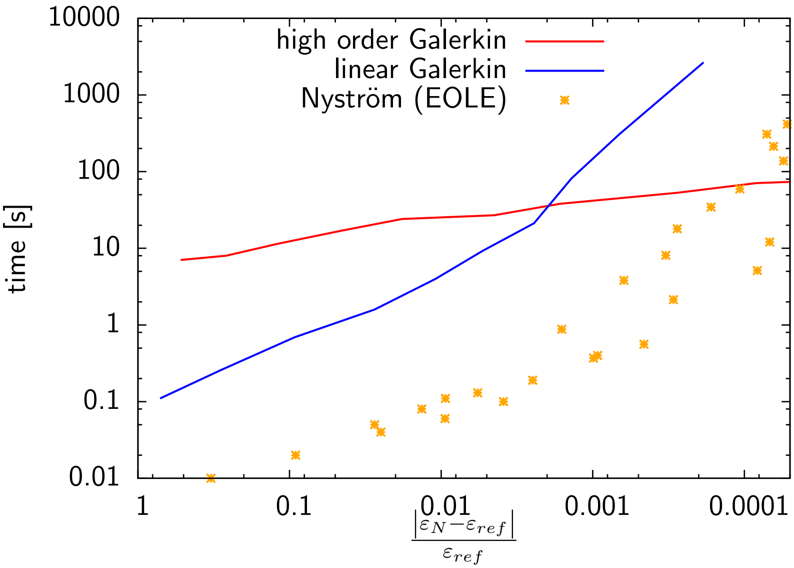
B: $\rho(\mathbf{x},\mathbf{x}') = \exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|}{1.08}\right)$

C: $\rho(\mathbf{x},\mathbf{x}') = \frac{1}{1+\left(\frac{|\mathbf{x}-\mathbf{x}'|}{0.725}\right)^{1.2}}$

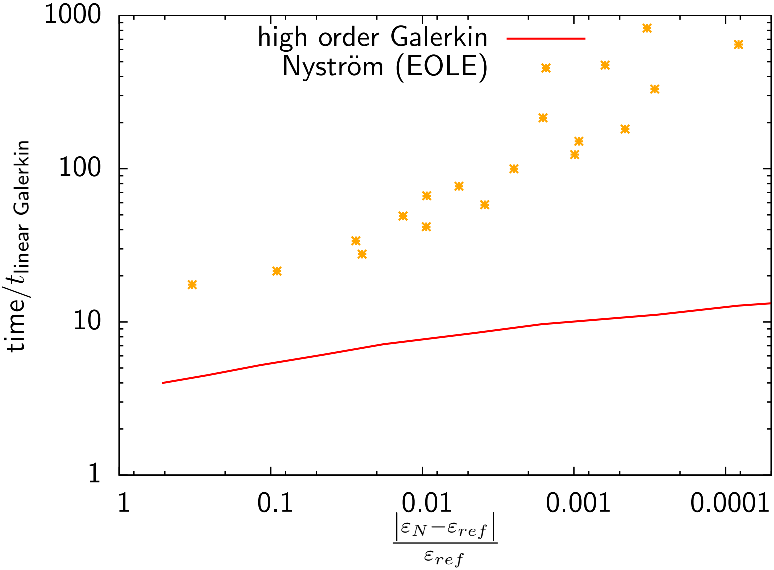
where $\varepsilon_{\text{ref}} \approx 10\%$



$M = 100$: time to converge (type A)



$M = 100$: time to compute a realization (type A)



Summary and Conclusion

Disadvantages

- **Nyström method**
  - **realization** expensive to evaluate
- **Galerkin method**
  - **implementation** is difficult
  - **solution** of the random field approximation is computationally expensive to obtain

Advantages

- **Nyström method**
  - straightforward **implementation**
  - fast **solution** of the random field approximation
- **Galerkin method**
  - realization computationally **cheap to evaluate**

Unsolved Problem (Unsolved Question) ???

Spectral stochastic finite elements

$$\mathbf{K}(\theta)\mathbf{U}(\theta) = \mathbf{F}(\theta) \tag{32}$$

Solution Vector

$$\mathbf{U}(\theta) = \sum_{j=0}^{P-1} u_j \cdot \Phi_j\left(\xi(\theta)\right) \tag{33}$$

- Galerkin approach to minimize $\mathbf{K}(\theta)\mathbf{U}(\theta) - \mathbf{F}(\theta) = \varepsilon$

Size of the problem to solve

$$N_{FEM} \cdot P \quad \text{with} \quad P = \sum_{k=0}^{p} \binom{M+k-1}{k} \tag{34}$$