Discretization of Random Fields Based on the Karhunen-Loève Expansion Using the Finite Cell Method

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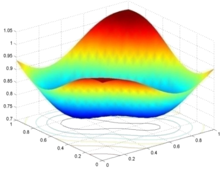
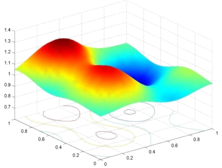
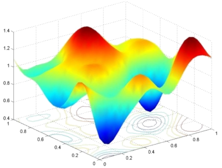
Motivation



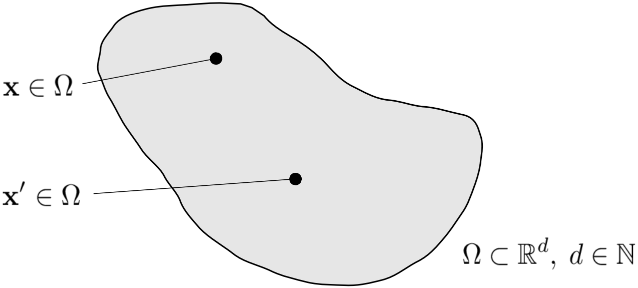
Application of random fields - Examples:

- soil properties in geotechnical engineering
- groundwater heights
- rainfall





Notation



- random field (RF): $H(\mathbf{x})$
- *Gaussian* random field - completely described by:
  - mean function $\mu(\mathbf{x})$
  - covariance function $\mathrm{Cov}(\mathbf{x}, \mathbf{x}') = \sigma(\mathbf{x}) \cdot \sigma(\mathbf{x}') \cdot \rho(\mathbf{x}, \mathbf{x}')$
    - $\sigma(\mathbf{x})$: standard deviation function
    - $\rho(\mathbf{x}, \mathbf{x}')$: correlation coefficient function

Random field discretization

Number of random variables (RVs) in a random field

- theoretically: *infinite* number of RVs (∞)
  - for each $\mathbf{x} \in \Omega$, $H(\mathbf{x})$ represents a RV
- discretized RF: *finite* number of RVs (M)

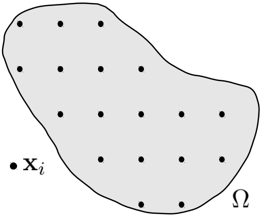
$$H(\mathbf{x}) \xrightarrow{\text{discretization}} \hat{H}(\mathbf{x}) \quad (1)$$

Categories of RF-discretization methods

- point discretization methods
- averaging discretization methods
- **series expansion methods**
  - Karhunen-Loève (KL) expansion
  - EOLE method

EOLE method - basic idea

- model a RV χ_i at each $\mathbf{x}_i$
  - $(\boldsymbol{\Sigma}_{\chi\chi})_{nm} = \mathrm{Cov}(\mathbf{x}_n, \mathbf{x}_m)$
- solve eigenvalue problem:
  - $\boldsymbol{\Sigma}_{\chi\chi}\boldsymbol{\Phi}_i = \theta_i \boldsymbol{\Phi}_i$ (for M largest θ_i)
- $\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\theta_i} h_i(\mathbf{x}) \xi_i$
  - $h_i(\mathbf{x}) = \boldsymbol{\Phi}_i^T \mathbf{b}(\mathbf{x})$
- find $\mathbf{b}^T(\mathbf{x})$ such that
  - minimize $\mathrm{Var}[\varepsilon_H(\mathbf{x})]$ subjected to $\mathrm{E}[\varepsilon_H] = 0$



EOLE-expansion

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \frac{\boldsymbol{\Phi}_i^T \boldsymbol{\Sigma}_{\chi\mathbf{x}}(\mathbf{x})}{\sqrt{\theta_i}} \xi_i \quad (2)$$

- with $(\boldsymbol{\Sigma}_{\chi\mathbf{x}}(\mathbf{x}))_j = \mathrm{Cov}(\mathbf{x}_j, \mathbf{x})$

EOLE method

EOLE-expansion

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \frac{\mathbf{\Phi}_i^T \mathbf{\Sigma}_{\chi\mathbf{x}}(\mathbf{x})}{\sqrt{\theta_i}} \xi_i \tag{3}$$

- $\left(\mathbf{\Sigma}_{\chi\mathbf{x}}(\mathbf{x})\right)_j = \mathrm{Cov}(\mathbf{x}_j, \mathbf{x})$
- solve $\mathbf{\Sigma}_{\chi\chi}\mathbf{\Phi}_i = \theta_i \mathbf{\Phi}_i$ with $\left(\mathbf{\Sigma}_{\chi\chi}\right)_{nm} = \mathrm{Cov}(\mathbf{x}_n, \mathbf{x}_m)$

- Note: geometry of Ω appears only *indirectly*

Karhunen-Loève expansion

KL-expansion

$$H(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, \varphi_i(\mathbf{x})\, \xi_i \tag{4}$$

- λ_i: eigenvalues of the covariance kernel
- φ_i: eigenfunctions of the covariance kernel
  - orthonormal: $\int_\Omega \varphi_i(\mathbf{x})\varphi_j(\mathbf{x})\,\mathrm{d}\mathbf{x} = \delta_{ij}$
- ξ_i: uncorrelated standard normal RVs
  - orthonormal: $\mathrm{E}[\xi_i \xi_j] = \delta_{ij}$

Integral eigenvalue problem

$$\int_{\mathbf{x}' \in \Omega} \varphi_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x}, \mathbf{x}')\,\mathrm{d}\mathbf{x}' = \lambda_i \varphi_i(\mathbf{x}) \tag{5}$$

Truncated KL-expansion

KL-expansion *(exact representation)*

$$H(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\,\varphi_i(\mathbf{x})\,\xi_i \tag{6}$$

Truncated KL-expansion *(approximation)*

$$\tilde{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\lambda_i}\,\varphi_i(\mathbf{x})\,\xi_i \tag{7}$$

- λ_i: M **largest** eigenvalues (in descending order)

Approximation of the KL-eigenfunctions

Integral eigenvalue problem (KL-expansion)

$$\int_{\mathbf{x}'\in\Omega} \varphi_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}' = \lambda_i\varphi_i(\mathbf{x}) \tag{8}$$

Approximation of the eigenfunctions

$$\hat{\varphi}_i(\mathbf{x}) = \sum_{n=1}^{N} d_n^i N_n(\mathbf{x}) = \mathbf{d}_i^T\,\mathbf{N}(\mathbf{x}) \tag{9}$$

- with $N_n(\mathbf{x}) \in L^2(\Omega)$
- Remember - EOLE: $h_i(\mathbf{x}) = \mathbf{\Phi}_i^T\mathbf{b}(\mathbf{x})$; $\mathbf{\Phi}_i$ known *a priori*.

Minimization of the resulting error

Approximated integral eigenvalue problem

$$\int_{\mathbf{x}' \in \Omega} \hat{\varphi}_i(\mathbf{x}') \mathrm{Cov}(\mathbf{x}, \mathbf{x}') \, \mathrm{d}\mathbf{x}' - \hat{\lambda}_i \hat{\varphi}_i(\mathbf{x}) = \tilde{\varepsilon}_N^i(\mathbf{x}) \tag{10}$$

Minimization of the resulting error (Galerkin)

$$\int_\Omega \tilde{\varepsilon}_N^i(\mathbf{x}) N_k(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 0 \tag{11}$$

Matrix eigenvalue problem

Matrix eigenvalue problem

$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \tag{12}$$

where

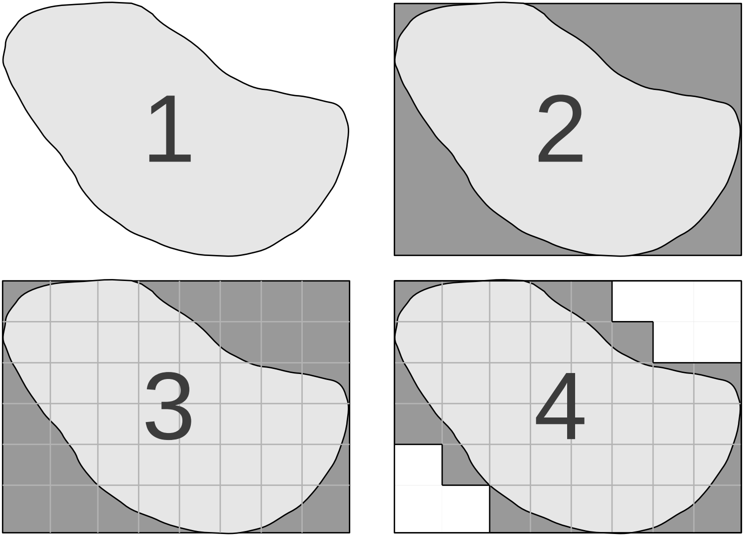
$$B_{kn} = \int_{\mathbf{x}\in\Omega} N_k(\mathbf{x}) \int_{\mathbf{x}'\in\Omega} N_n(\mathbf{x}')\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \tag{13}$$

$$M_{ij} = \int_{\mathbf{x}\in\Omega} N_i(\mathbf{x}) N_j(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{14}$$

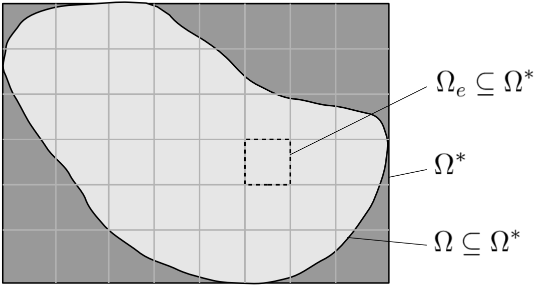
Approximated truncated KL-expansion

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\hat{\lambda}_i}\,\hat{\varphi}_i(\mathbf{x})\,\xi_i \tag{15}$$

Finite cell - basic idea



Finite cell - notation



- global shape functions: $N_i \in L^2(\Omega^*)$

$$\alpha(\mathbf{x}) = \begin{cases} 1 & \forall \mathbf{x} \in \Omega \\ 0 & \forall \mathbf{x} \in \Omega^* \setminus \Omega \end{cases} \tag{16}$$

Finite cell approach of the pFEM-KL-expansion

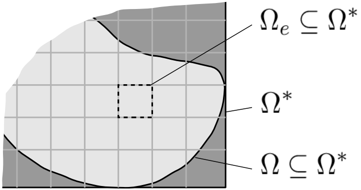
Matrix eigenvalue problem

$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \quad (17)$$

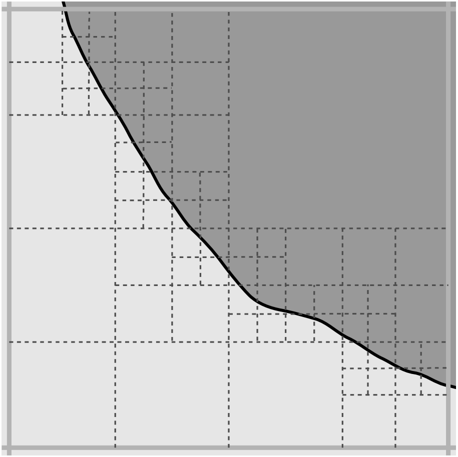
where

$$B_{kn} = \int_{\mathbf{x}\in\Omega^*} \alpha(\mathbf{x}) N_k(\mathbf{x}) \int_{\mathbf{x}'\in\Omega^*} \alpha(\mathbf{x}') N_n(\mathbf{x}') \mathrm{Cov}(\mathbf{x},\mathbf{x}') \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \quad (18)$$

$$M_{ij} = \int_{\mathbf{x}\in\Omega^*} \alpha(\mathbf{x}) N_i(\mathbf{x}) N_j(\mathbf{x}) \, \mathrm{d}\mathbf{x} \quad (19)$$



Staggered Gaussian integration



Error variance

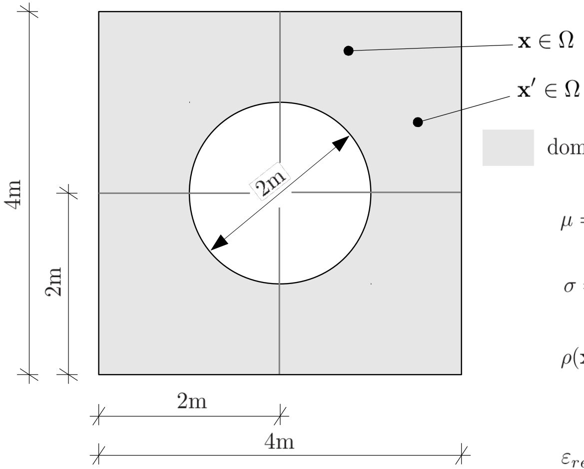
Error variance

$$\varepsilon_\sigma(\mathbf{x}) = \frac{\mathrm{Var}\left[H(\mathbf{x}) - \hat{H}(\mathbf{x})\right]}{\sigma^2(\mathbf{x})} \tag{20}$$

Mean error variance

$$\varepsilon = \frac{\int_\Omega \varepsilon_\sigma(\mathbf{x})\,\mathrm{d}\mathbf{x}}{|\Omega|} \tag{21}$$

Example of a plate with a hole



domain Ω of the random field with

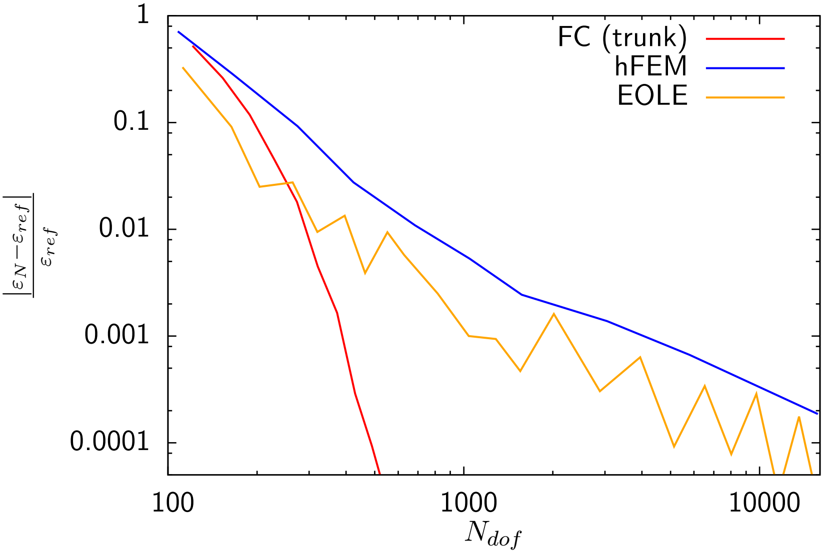
$$\mu = 30 \cdot 10^3 \frac{\mathrm{MN}}{\mathrm{m}^2}$$

$$\sigma = 6 \cdot 10^3 \frac{\mathrm{MN}}{\mathrm{m}^2}$$

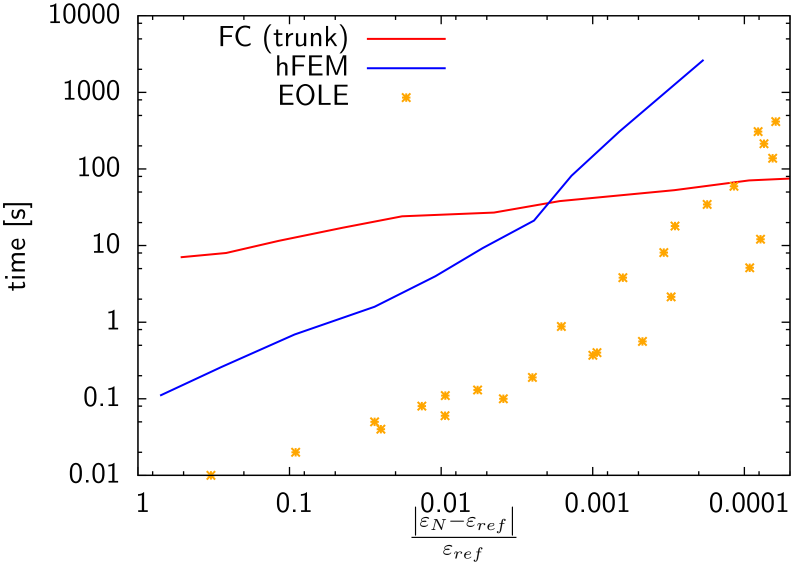
$$\rho(\mathbf{x}, \mathbf{x}') = \exp\left(-\left(\frac{|\mathbf{x} - \mathbf{x}'|}{0.3325}\right)^2\right)$$

$$\varepsilon_{ref} = 0.09978$$

$M = 100$: relative error



$M = 100$: time


10000
1000
100
10
1
0.1
0.01
time [s]
FC (trunk)
hFEM
EOLE
1
0.1
0.01
0.001
0.0001
$\frac{|\varepsilon_N - \varepsilon_{ref}|}{\varepsilon_{ref}}$

Summary and Conclusion

FC-KL - Cons

- Computationally very **expensive to solve** *(especially in 3D)*
  - FC-KL: double integral over covariance function
  - EOLE: just $N \times N$ covariance function (and Lanczos methods)
- Quite **difficult to implement** *(compared to EOLE)*
  - pFEM
  - (double) integration of non-continuous non-smooth functions
- **Numerical stability** of eigenvalue problem

FC-KL - Pros

- **Fast convergence** against optimal representation
- Realization computationally **cheap to evaluate**
  - **Efficient assembly of FE stiffness matrices**
- **Simple mesh**