A finite cell approach for discretization of random fields

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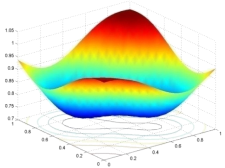
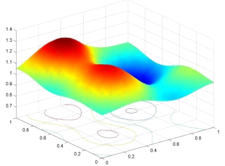
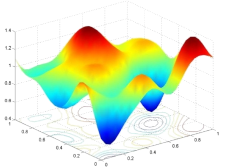
Motivation

Application of random fields - Examples:

- soil properties in geotechnical engineering
- groundwater heights
- rainfall

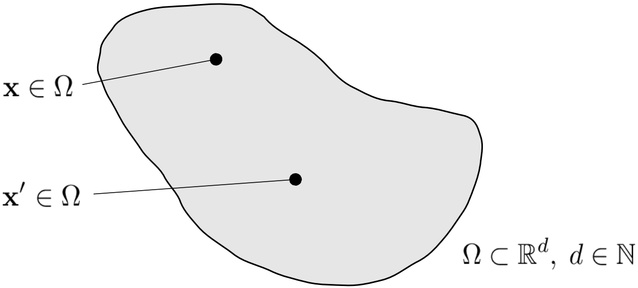






1. Introduction
    - Random fields
    - Karhunen-Loève (KL) expansion
2. Finite cell based KL-approximation
    - Higher order FEM discretization
    - Finite cell based discretization
3. Numerical Studies
    - 2D-Example
4. Summary

Notation



- random field (RF): $H(\mathbf{x})$
- *Gaussian* random field - completely described by:
  - mean function $\mu(\mathbf{x})$
  - covariance function $\mathrm{Cov}(\mathbf{x}, \mathbf{x}') = \sigma(\mathbf{x}) \cdot \sigma(\mathbf{x}') \cdot \rho(\mathbf{x}, \mathbf{x}')$
    - $\sigma(\mathbf{x})$: standard deviation function
    - $\rho(\mathbf{x}, \mathbf{x}')$: correlation coefficient function

Random field discretization

Number of random variables (RVs) in a random field

- theoretically: *infinite* number of RVs (∞)
  - for each $\mathbf{x} \in \Omega$, $H(\mathbf{x})$ represents a RV
- discretized RF: *finite* number of RVs (M)

$$H(\mathbf{x}) \xrightarrow{\text{discretization}} \hat{H}(\mathbf{x}) \tag{1}$$

Categories of RF-discretization methods

- point discretization methods
- averaging discretization methods
- **series expansion methods**
  - Karhunen-Loève (KL) expansion
  - EOLE method

Karhunen-Loève (KL) expansion

(truncated) KL-expansion

$$H(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty\ M} \sqrt{\lambda_i}\,\varphi_i(\mathbf{x})\,\xi_i \tag{2}$$

- λ_i: (M largest) eigenvalues of the covariance kernel
- φ_i: eigenfunctions of the covariance kernel
  - orthonormal: $\int_\Omega \varphi_i(\mathbf{x})\varphi_j(\mathbf{x})\,\mathrm{d}\mathbf{x} = \delta_{ij}$
- ξ_i: uncorrelated standard normal RVs
  - orthonormal: $\mathrm{E}[\xi_i \xi_j] = \delta_{ij}$

Integral eigenvalue problem

$$\int_{\mathbf{x}' \in \Omega} \varphi_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x}, \mathbf{x}')\,\mathrm{d}\mathbf{x}' = \lambda_i \varphi_i(\mathbf{x}) \tag{3}$$

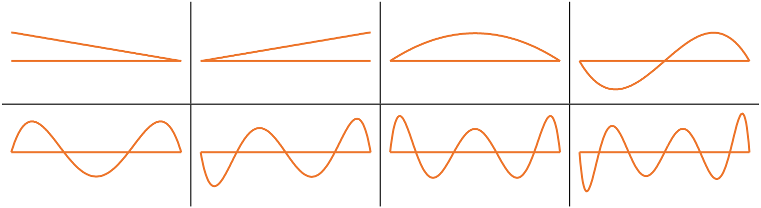
Approximation of the KL-eigenfunctions

Integral eigenvalue problem (KL-expansion)

$$\int_{\mathbf{x}'\in\Omega} \varphi_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}' = \lambda_i\varphi_i(\mathbf{x}) \tag{4}$$

Approximation of the eigenfunctions

$$\hat{\varphi}_i(\mathbf{x}) = \sum_{n=1}^{N} d_n^i N_n(\mathbf{x}) = \mathbf{d}_i^T\,\mathbf{N}(\mathbf{x}); \quad N_n(\mathbf{x}) \in L^2(\Omega) \tag{5}$$



Matrix eigenvalue problem

Matrix eigenvalue problem

$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \tag{6}$$

where

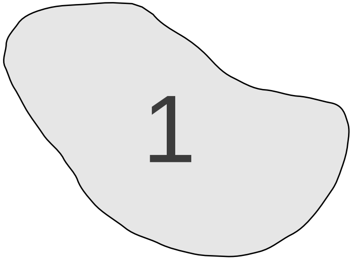
$$B_{kn} = \int\limits_{\mathbf{x}\in\Omega} N_k(\mathbf{x}) \int\limits_{\mathbf{x}'\in\Omega} N_n(\mathbf{x}') \mathrm{Cov}(\mathbf{x},\mathbf{x}') \,\mathrm{d}\mathbf{x}' \,\mathrm{d}\mathbf{x} \tag{7}$$

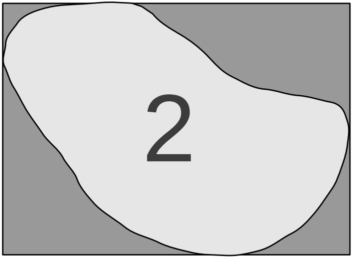
$$M_{ij} = \int_{\mathbf{x}\in\Omega} N_i(\mathbf{x}) N_j(\mathbf{x}) \,\mathrm{d}\mathbf{x} \tag{8}$$

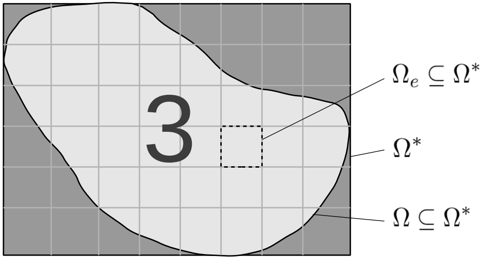
Approximated truncated KL-expansion

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\hat{\lambda}_i}\, \hat{\varphi}_i(\mathbf{x})\, \xi_i \tag{9}$$

Finite cell - basic idea & notation







- global shape functions: $N_i \in L^2(\Omega^*)$
- indicator function:

$$\alpha(\mathbf{x}) = \begin{cases} 1 & \forall\, \mathbf{x} \in \Omega \\ 0 & \forall\, \mathbf{x} \in \Omega^* \setminus \Omega \end{cases}$$

Finite cell approach of the pFEM-KL-expansion

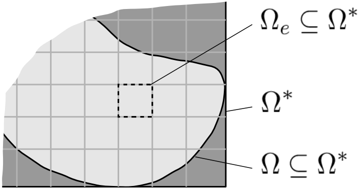
Matrix eigenvalue problem

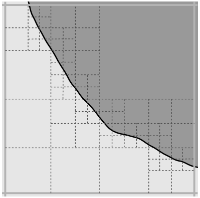
$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \tag{10}$$

where

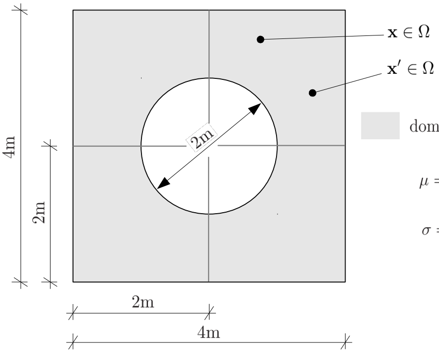
$$B_{kn} = \int_{\mathbf{x}\in\Omega^*} \alpha(\mathbf{x}) N_k(\mathbf{x}) \int_{\mathbf{x}'\in\Omega^*} \alpha(\mathbf{x}') N_n(\mathbf{x}') \mathrm{Cov}(\mathbf{x},\mathbf{x}') \,\mathrm{d}\mathbf{x}' \,\mathrm{d}\mathbf{x} \tag{11}$$

$$M_{ij} = \int_{\mathbf{x}\in\Omega^*} \alpha(\mathbf{x}) N_i(\mathbf{x}) N_j(\mathbf{x}) \,\mathrm{d}\mathbf{x} \tag{12}$$





Example of a plate with a hole



domain Ω of the random field with

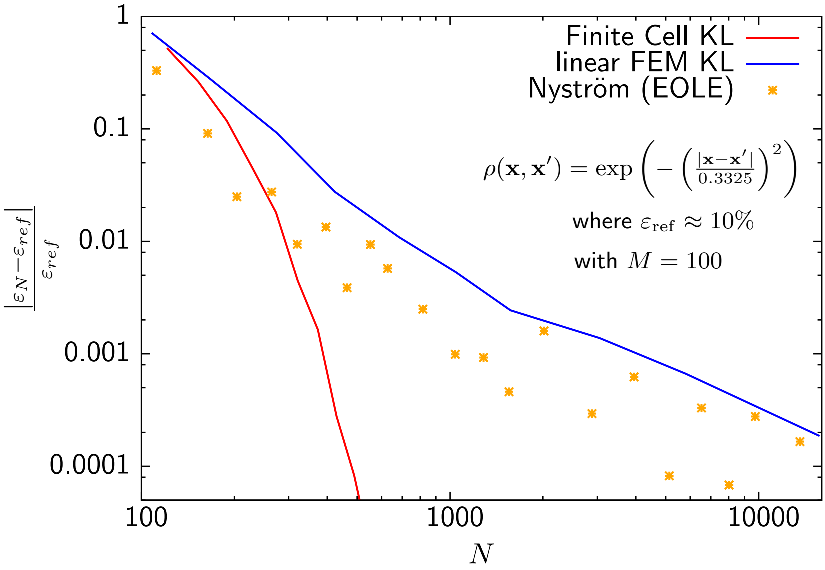
$$\mu = 30 \cdot 10^3 \frac{\mathrm{MN}}{\mathrm{m}^2}$$

$$\sigma = 6 \cdot 10^3 \frac{\mathrm{MN}}{\mathrm{m}^2}$$

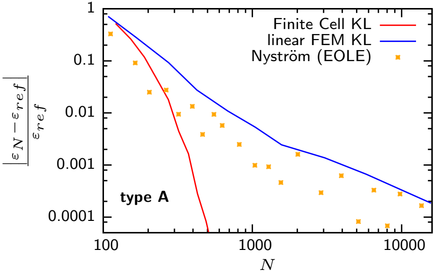
Investigated discretization methods

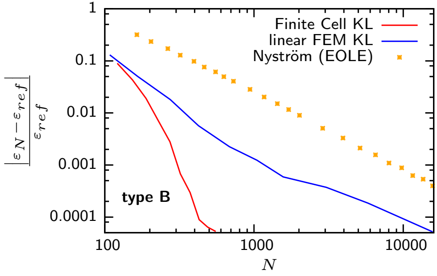
- **Finite Cell KL** (4 cells; increase in polynomial order)
- **linear FEM KL** (linear shape functions; increase number of elements)
- **Nyström (EOLE)** (icrease number of points)

1^{st} study: convergence of relative error w.r.t. N



1st study: convergence of relative error w.r.t. N



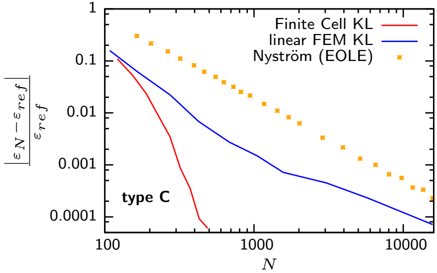


A: $\rho(\mathbf{x},\mathbf{x}') = \exp\left(-\left(\frac{|\mathbf{x}-\mathbf{x}'|}{0.3325}\right)^2\right)$

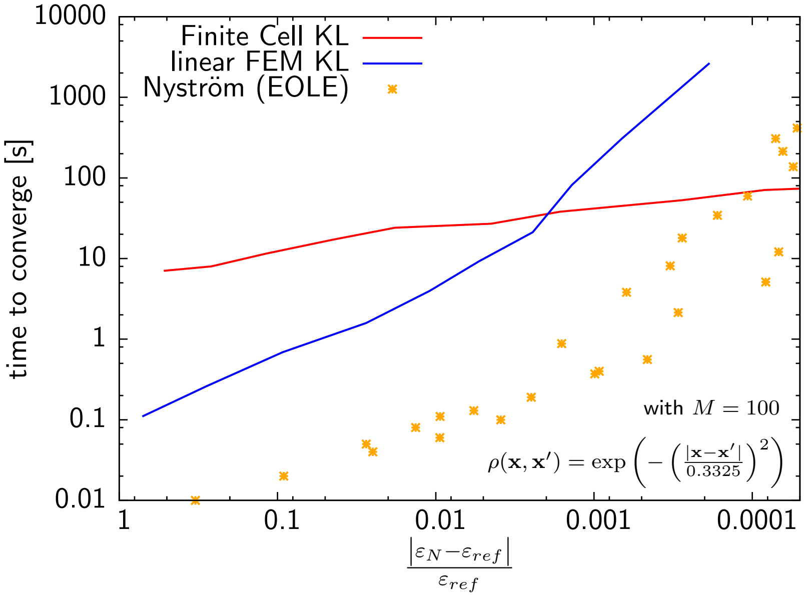
B: $\rho(\mathbf{x},\mathbf{x}') = \exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|}{1.08}\right)$

C: $\rho(\mathbf{x},\mathbf{x}') = \frac{1}{1+\left(\frac{|\mathbf{x}-\mathbf{x}'|}{0.725}\right)^{1.2}}$

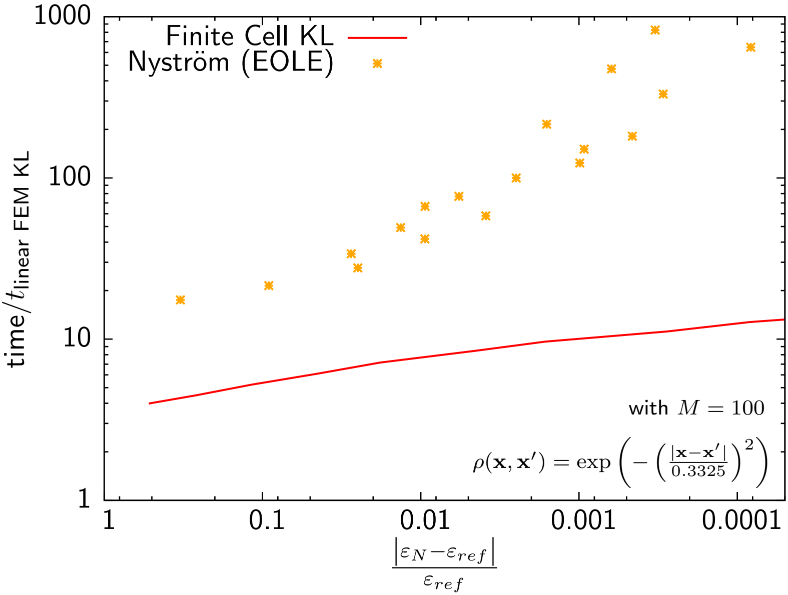
where $\varepsilon_{\text{ref}} \approx 10\%$



2nd study: time needed to obtain a random field approx.



3rd study: time needed to compute a realization



Summary and Conclusion

FC-KL - Cons

- Computationally very **expensive to solve** *(especially in 3D)*
  - FC-KL: double integral over covariance function
  - EOLE: just $N \times N$ covariance function (and Lanczos methods)
- Quite **difficult to implement** *(compared to EOLE)*
  - higher order shape functions
  - (double) integration of non-continuous non-smooth functions
- **Numerical stability** of eigenvalue problem

FC-KL - Pros

- **Fast convergence** against optimal representation
- Realization computationally **cheap to evaluate**
  - **Efficient assembly of FE stiffness matrices**
- **Simple mesh**