Quasi meshless discretization of random fields based on the Karhunen-Loève expansion

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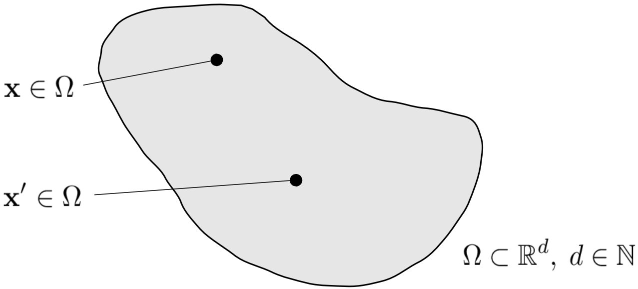
2012-02-01

Notation



- random field (RF): $H(\mathbf{x})$
- *Gaussian* random field - completely described by:
  - mean function $\mu(\mathbf{x})$
  - covariance function $\mathrm{Cov}(\mathbf{x}, \mathbf{x}') = \sigma(\mathbf{x}) \cdot \sigma(\mathbf{x}') \cdot \rho(\mathbf{x}, \mathbf{x}')$
    - $\sigma(\mathbf{x})$: standard deviation function
    - $\rho(\mathbf{x}, \mathbf{x}')$: correlation coefficient function

Random field discretization

Number of random variables (RVs) in a random field

- theoretically: *infinite* number of RVs (∞)
  - for each $\mathbf{x} \in \Omega$, $H(\mathbf{x})$ represents a RV
- discretized RF: *finite* number of RVs (M)

$$H(\mathbf{x}) \xrightarrow{\text{discretization}} \hat{H}(\mathbf{x}) \tag{1}$$

Categories of RF-discretization methods

- point discretization methods
- averaging discretization methods
- **series expansion methods**
  - Karhunen-Loève (KL) expansion
  - EOLE method

Mean square error

Truncation error

$$\varepsilon_H(\mathbf{x}) = H(\mathbf{x}) - \hat{H}(\mathbf{x}) \tag{2}$$

- Assumption: $\mathrm{E}\left[H(\mathbf{x})\right] = \mathrm{E}\left[\hat{H}(\mathbf{x})\right] \Rightarrow \mathrm{E}\left[\varepsilon_H(\mathbf{x})\right] = 0$

Mean square error

$$\mathrm{E}\left[\varepsilon_H^2(\mathbf{x})\right] = \mathrm{Var}\left[\varepsilon_H(\mathbf{x})\right] = \mathrm{E}\left[\left(H(\mathbf{x}) - \hat{H}(\mathbf{x})\right)^2\right] \tag{3}$$

- $\mathrm{E}\left[\varepsilon_H^2(\mathbf{x})\right] = \mathrm{Var}\left[H(\mathbf{x})\right] + \mathrm{Var}\left[\hat{H}(\mathbf{x})\right] - 2 \cdot \mathrm{Cov}\left[H(\mathbf{x}), \hat{H}(\mathbf{x})\right]$

Global mean square error

$$\bar{\varepsilon}_H^2(\mathbf{x}) = \int_\Omega \mathrm{E}\left[\varepsilon_H^2(\mathbf{x})\right] \, \mathrm{d}\mathbf{x} \tag{4}$$

Error variance

Error variance

$$\varepsilon_\sigma(\mathbf{x}) = \frac{\mathrm{Var}\left[H(\mathbf{x}) - \hat{H}(\mathbf{x})\right]}{\sigma^2(\mathbf{x})} \tag{5}$$

- $\varepsilon_\sigma(\mathbf{x}) = \mathrm{E}\left[\varepsilon_H^2(\mathbf{x})\right] / \sigma^2(\mathbf{x})$

Mean error variance

$$\bar{\varepsilon}_\sigma = \frac{\int_\Omega \varepsilon_\sigma(\mathbf{x})\,\mathrm{d}\mathbf{x}}{|\Omega|} \tag{6}$$

Supremum norm of the error variance

$$\hat{\varepsilon}_\sigma = \sup_{\mathbf{x}\in\Omega} \varepsilon_\sigma \tag{7}$$

Covariance error

Covariance error

$$\varepsilon_\rho(\mathbf{x}) = \frac{\int_{\mathbf{x}'\in\Omega} \left| \mathrm{Cov}\left(H(\mathbf{x}), H(\mathbf{x}')\right) - \mathrm{Cov}\left(\hat{H}(\mathbf{x}), \hat{H}(\mathbf{x}')\right) \right| \,\mathrm{d}\mathbf{x}'}{\int_{\mathbf{x}'\in\Omega} \left| \mathrm{Cov}\left(H(\mathbf{x}), H(\mathbf{x}')\right) \right| \,\mathrm{d}\mathbf{x}'} \tag{8}$$

Mean covariance error

$$\overline{\varepsilon}_\rho = \frac{\int_\Omega \varepsilon_\rho(\mathbf{x})\,\mathrm{d}\mathbf{x}}{|\Omega|} \tag{9}$$

Supremum norm of the covariance error

$$\hat{\varepsilon}_\rho = \sup_{\mathbf{x}\in\Omega} \varepsilon_\rho \tag{10}$$

Karhunen-Loève expansion

KL-expansion

$$H(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, \varphi_i(\mathbf{x})\, \xi_i \tag{11}$$

- λ_i: eigenvalues of the covariance kernel
- φ_i: eigenfunctions of the covariance kernel
  - orthonormal: $\int_\Omega \varphi_i(\mathbf{x})\varphi_j(\mathbf{x})\,\mathrm{d}\mathbf{x} = \delta_{ij}$
- ξ_i: uncorrelated standard normal RVs
  - orthonormal: $\mathrm{E}[\xi_i \xi_j] = \delta_{ij}$

Integral eigenvalue problem

$$\int\limits_{\mathbf{x}' \in \Omega} \varphi_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x}, \mathbf{x}')\,\mathrm{d}\mathbf{x}' = \lambda_i \varphi_i(\mathbf{x}) \tag{12}$$

Truncated KL-expansion

Truncated KL-expansion

$$\tilde{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\lambda_i}\, \varphi_i(\mathbf{x})\, \xi_i \tag{13}$$

- λ_i: M **largest** eigenvalues (in descending order)

Variance

$$\tilde{\mathrm{Var}}(\mathbf{x}) = \sum_{i=1}^{M} \lambda_i\, \varphi_i^2(\mathbf{x}) \tag{14}$$

Covariance

$$\tilde{\mathrm{Cov}}(\mathbf{x}, \mathbf{x}') = \sum_{i=1}^{M} \lambda_i\, \varphi_i(\mathbf{x})\, \varphi_i(\mathbf{x}') \tag{15}$$

KL-truncation error

Mean square error

$$\mathrm{E}\left[\varepsilon_H^2(\mathbf{x})\right] = \sigma^2(\mathbf{x}) - \sum_{i=1}^{M} \lambda_i\, \varphi_i^2(\mathbf{x}) \tag{16}$$

- **optimal** w.r.t. $\int_\Omega \mathrm{Var}\left[\varepsilon_H(\mathbf{x})\right] \mathrm{d}\mathbf{x} = \sum_{i=M+1}^{\infty} \lambda_i$

Error variance

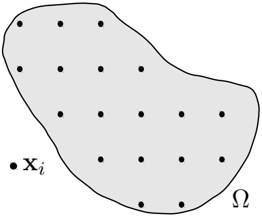
$$\varepsilon_\sigma(\mathbf{x}) = 1 - \frac{1}{\sigma^2(\mathbf{x})} \sum_{i=1}^{M} \lambda_i\, \varphi_i^2(\mathbf{x}) \tag{17}$$

Mean error variance

$$\overline{\varepsilon}_\sigma = 1 - \frac{1}{\sigma^2} \sum_{i=1}^{M} \lambda_i \quad \text{with } \sigma = \sigma(\mathbf{x}) = \text{const.} \tag{18}$$

EOLE method - basic idea

- model a RV χ_i at each $\mathbf{x}_i$
  - $(\boldsymbol{\Sigma}_{\chi\chi})_{nm} = \mathrm{Cov}(\mathbf{x}_n, \mathbf{x}_m)$
- solve eigenvalue problem:
  - $\boldsymbol{\Sigma}_{\chi\chi}\boldsymbol{\Phi}_i = \theta_i \boldsymbol{\Phi}_i$ (for M largest θ_i)
- $\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\theta_i} h_i(\mathbf{x}) \xi_i$
  - $h_i(\mathbf{x}) = \boldsymbol{\Phi}_i^T \mathbf{b}(\mathbf{x})$
- find $\mathbf{b}^T(\mathbf{x})$ such that
  - minimize $\mathrm{Var}[\varepsilon_H(\mathbf{x})]$ subjected to $\mathrm{E}[\varepsilon_H] = 0$



EOLE-expansion

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \frac{\boldsymbol{\Phi}_i^T \boldsymbol{\Sigma}_{\chi\mathbf{x}}(\mathbf{x})}{\sqrt{\theta_i}} \xi_i \qquad (19)$$

- with $(\boldsymbol{\Sigma}_{\chi\mathbf{x}}(\mathbf{x}))_j = \mathrm{Cov}(\mathbf{x}_j, \mathbf{x})$

EOLE method

EOLE-expansion

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \frac{\mathbf{\Phi}_i^T \mathbf{\Sigma}_{\chi\mathbf{x}}(\mathbf{x})}{\sqrt{\theta_i}}\, \xi_i \qquad (20)$$

- $\left(\mathbf{\Sigma}_{\chi\mathbf{x}}(\mathbf{x})\right)_j = \mathrm{Cov}(\mathbf{x}_j, \mathbf{x})$
- solve $\mathbf{\Sigma}_{\chi\chi}\mathbf{\Phi}_i = \theta_i \mathbf{\Phi}_i$ with $\left(\mathbf{\Sigma}_{\chi\chi}\right)_{nm} = \mathrm{Cov}(\mathbf{x}_n, \mathbf{x}_m)$

Error variance

$$\varepsilon_\sigma(\mathbf{x}) = 1 - \frac{1}{\sigma^2(\mathbf{x})} \sum_{i=1}^{M} \frac{\left(\mathbf{\Phi}_i^T \mathbf{\Sigma}_{\chi\mathbf{x}}(\mathbf{x})\right)^2}{\theta_i} \qquad (21)$$

- Note: geometry of Ω appears only *indirectly*

Approximation of the KL-eigenfunctions

Integral eigenvalue problem (KL-expansion)

$$\int_{\mathbf{x}'\in\Omega} \varphi_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}' = \lambda_i\varphi_i(\mathbf{x}) \tag{22}$$

Approximation of the eigenfunctions

$$\hat{\varphi}_i(\mathbf{x}) = \sum_{n=1}^{N} d_n^i N_n(\mathbf{x}) = \mathbf{d}_i^T\,\mathbf{N}(\mathbf{x}) \tag{23}$$

- with $N_n(\mathbf{x}) \in L^2(\Omega)$
- Remember - EOLE: $h_i(\mathbf{x}) = \mathbf{\Phi}_i^T\mathbf{b}(\mathbf{x})$; $\mathbf{\Phi}_i$ known *a priori*.

Minimization of the resulting error

Approximated integral eigenvalue problem

$$\int_{\mathbf{x}'\in\Omega} \hat{\varphi}_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}' - \hat{\lambda}_i\hat{\varphi}_i(\mathbf{x}) = \tilde{\varepsilon}_N^i(\mathbf{x}) \tag{24}$$

Minimization of the resulting error (Galerkin)

$$\int_\Omega \tilde{\varepsilon}_N^i(\mathbf{x})N_k(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \tag{25}$$

Matrix eigenvalue problem

Matrix eigenvalue problem

$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \tag{26}$$

where

$$B_{kn} = \int\limits_{\mathbf{x}\in\Omega} N_k(\mathbf{x}) \int\limits_{\mathbf{x}'\in\Omega} N_n(\mathbf{x}')\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \tag{27}$$

$$M_{ij} = \int_{\mathbf{x}\in\Omega} N_i(\mathbf{x})N_j(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{28}$$

Approximated truncated KL-expansion

Approximated truncated KL-expansion

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\hat{\lambda}_i}\, \hat{\varphi}_i(\mathbf{x})\, \xi_i \qquad (29)$$

Error variance

$$\varepsilon_\sigma(\mathbf{x}) = 1 + \frac{\sum_{i=1}^{M} \hat{\lambda}_i\, \hat{\varphi}_i^2(\mathbf{x}) - 2\sum_{i=1}^{M} \hat{\varphi}_i(\mathbf{x}) \int_\Omega \mathrm{Cov}(\mathbf{x}, \mathbf{x}')\hat{\varphi}_i(\mathbf{x}')\, \mathrm{d}\mathbf{x}'}{\sigma^2(\mathbf{x})} \qquad (30)$$

Mean error variance

$$\bar{\varepsilon}_\sigma = 1 - \frac{1}{\sigma^2} \sum_{i=1}^{M} \hat{\lambda}_i \quad \text{with } \sigma = \sigma(\mathbf{x}) = \text{const.} \qquad (31)$$

Optimality of KL, EOLE, approximated-KL

KL, *EOLE* and *approximated-KL* are based on a conditioned minimization of $\mathrm{Var}[\varepsilon_H(\mathbf{x})]$.

Karhunen-Loève

minimization *(globally)* w.r.t. any orthogonal basis of $L^2(\Omega)$

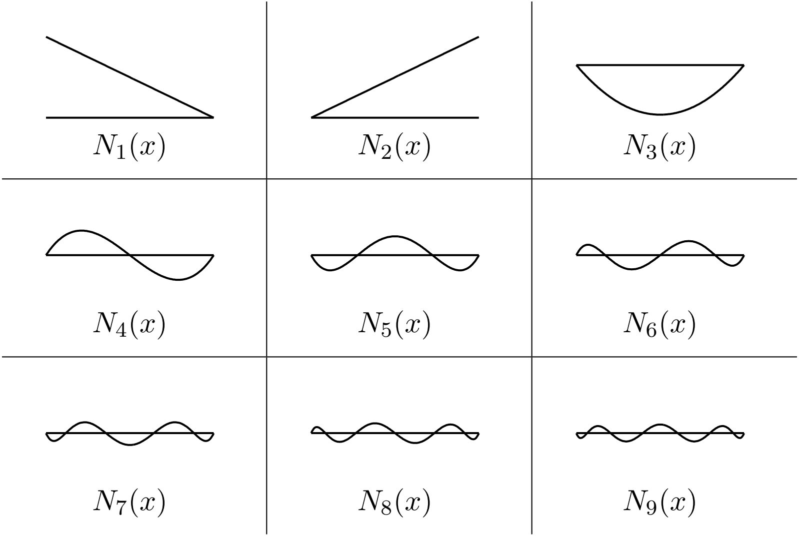
approximated-KL

minimization *(globally)* w.r.t. any orthogonal basis spanning the subspace of $L^2(\Omega)$ defined by $\{N_i(\mathbf{x})\}_{i=1}^{N}$

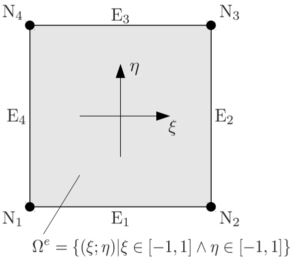
EOLE

minimization *(point-wise)* w.r.t. the chosen points $\{\mathbf{x}_i\}_{i=1}^{N}$

Hierarchic Legendre polynomials (1D)



2D hierarchic basis (nodal modes and edge modes)



- Nodal modes

$$N_{1,1}(\xi,\eta) = N_1(\xi)N_1(\eta) \quad \text{(32a)}$$
$$N_{2,1}(\xi,\eta) = N_2(\xi)N_1(\eta) \quad \text{(32b)}$$
$$N_{2,2}(\xi,\eta) = N_2(\xi)N_2(\eta) \quad \text{(32c)}$$
$$N_{1,2}(\xi,\eta) = N_1(\xi)N_2(\eta) \quad \text{(32d)}$$

- Edge modes

$$N_{i,1}(\xi,\eta) = N_i(\xi)N_1(\eta) \qquad i = 3,4,\ldots,p_{e1}+1 \quad \text{(33a)}$$
$$N_{2,i}(\xi,\eta) = N_2(\xi)N_i(\eta) \qquad i = 3,4,\ldots,p_{e2}+1 \quad \text{(33b)}$$
$$N_{i,2}(\xi,\eta) = N_i(\xi)N_2(\eta) \qquad i = 3,4,\ldots,p_{e3}+1 \quad \text{(33c)}$$
$$N_{1,i}(\xi,\eta) = N_1(\xi)N_i(\eta) \qquad i = 3,4,\ldots,p_{e4}+1 \quad \text{(33d)}$$

2D hierarchic basis (face modes)

Tensor space

$$N_{i+1,j+1}(\xi,\eta) = N_{i+1}(\xi)N_{j+1}(\eta) \\ i = 2,\ldots,p_\xi,\ j = 2,\ldots,p_\eta \qquad (34)$$

Trunk space

$$N_{i+1,j+1}(\xi,\eta) = N_{i+1}(\xi)N_{j+1}(\eta) \\ i = 2,\ldots,p_\xi,\ j = 2,\ldots,p_\eta, \\ i+j = 4,\ldots,\max\{p_\xi,p_\eta\} \qquad (35)$$

1D Example

Example

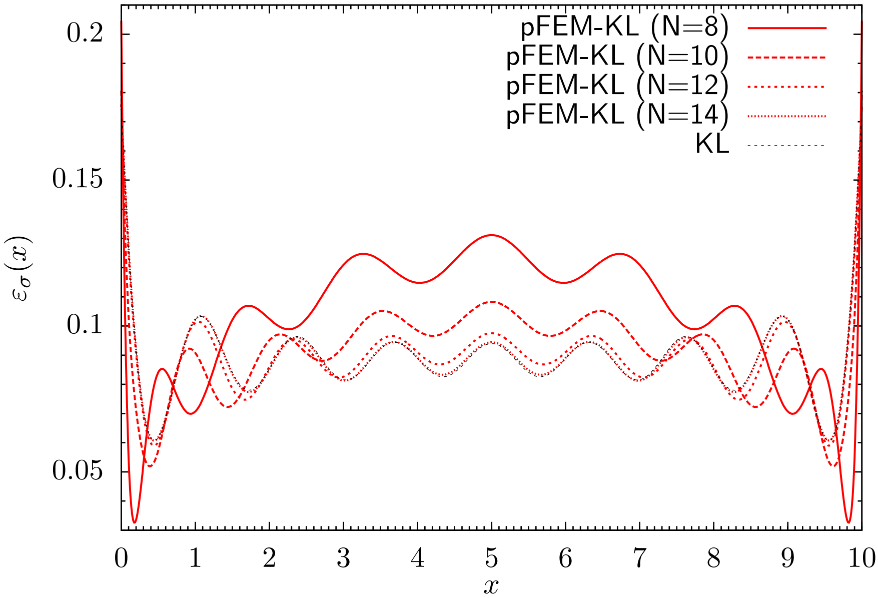
1D, straight domain

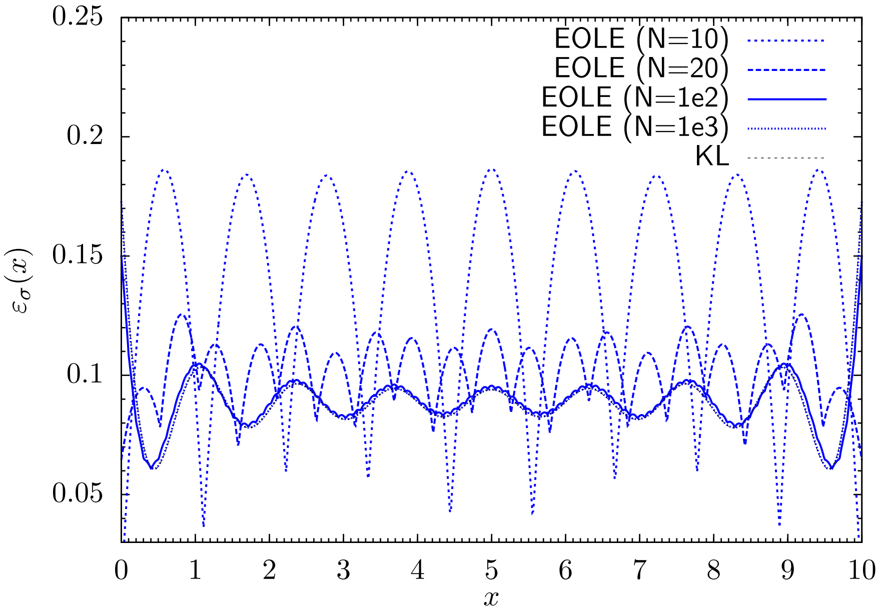
- length of domain: $l = 10$
- standard deviation: $\sigma = 1$
- correlation coefficient function: $\rho(x, x') = \exp\left(-\frac{|x-x'|}{a}\right)$
- correlation length: $a = 3$

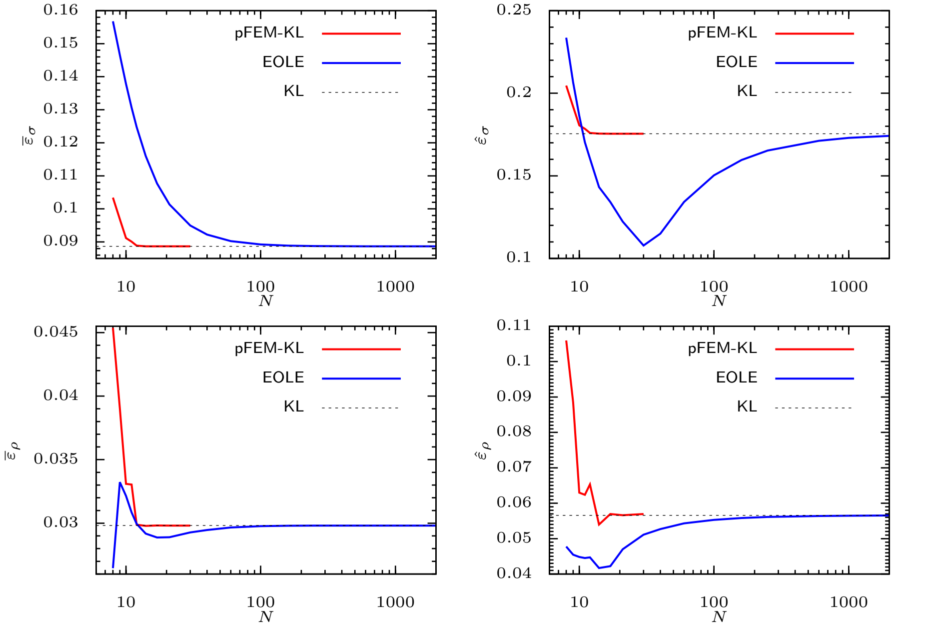
- pFEM-KL: just *one* element

Parameter study

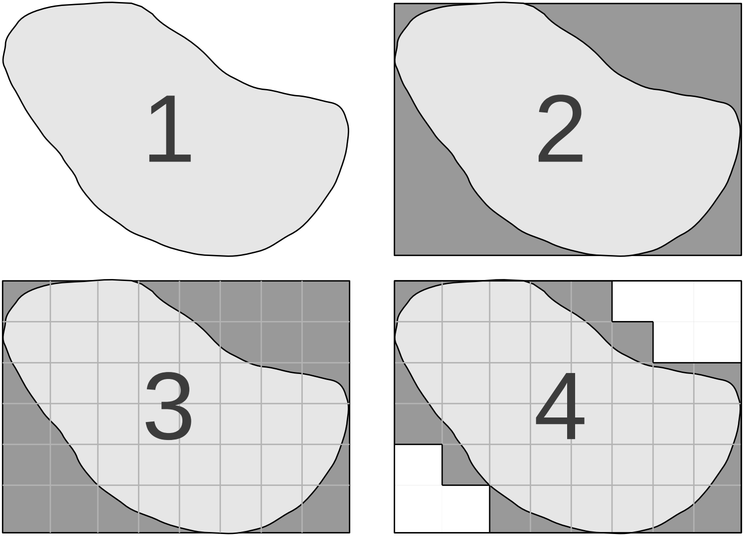
- **FIX** $M = 8$
- **MODIFY** N


pFEM-KL (N=8)
pFEM-KL (N=10)
pFEM-KL (N=12)
pFEM-KL (N=14)
KL
$\varepsilon_\sigma(x)$
x
0.05
0.1
0.15
0.2
0
1
2
3
4
5
6
7
8
9
10

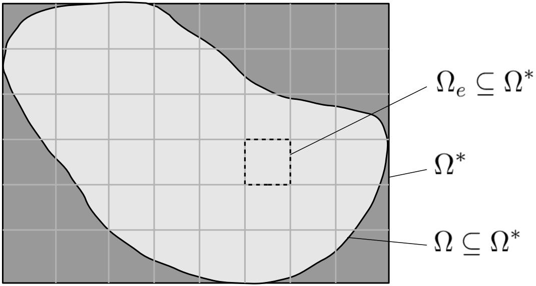

EOLE (N=10)
EOLE (N=20)
EOLE (N=1e2)
EOLE (N=1e3)
KL
0.25
0.2
0.15
0.1
0.05
$\varepsilon_\sigma(x)$
0
1
2
3
4
5
6
7
8
9
10
x


pFEM-KL
EOLE
KL
$\bar{\varepsilon}_\sigma$
0.16
0.15
0.14
0.13
0.12
0.11
0.1
0.09
$\hat{\varepsilon}_\sigma$
0.25
0.2
0.15
0.1
$\bar{\varepsilon}_\rho$
0.045
0.04
0.035
0.03
$\hat{\varepsilon}_\rho$
0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
10
100
1000
N

Finite cell - basic idea



Finite cell - notation



- global shape functions: $N_i \in L^2(\Omega^*)$

$$\alpha(\mathbf{x}) = \begin{cases} 1 & \forall \mathbf{x} \in \Omega \\ 0 & \forall \mathbf{x} \in \Omega^* \setminus \Omega \end{cases} \quad (36)$$

Finite cell approach of the pFEM-KL-expansion

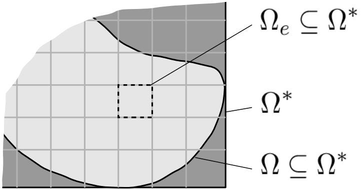
Matrix eigenvalue problem

$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \tag{37}$$

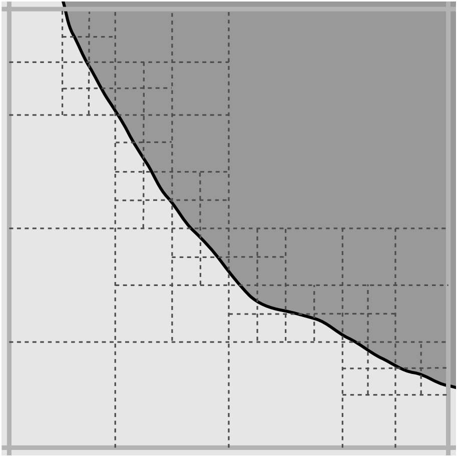
where

$$B_{kn} = \int_{\mathbf{x}\in\Omega^*} \alpha(\mathbf{x}) N_k(\mathbf{x}) \int_{\mathbf{x}'\in\Omega^*} \alpha(\mathbf{x}') N_n(\mathbf{x}') \mathrm{Cov}(\mathbf{x},\mathbf{x}')\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} \tag{38}$$

$$M_{ij} = \int_{\mathbf{x}\in\Omega^*} \alpha(\mathbf{x}) N_i(\mathbf{x}) N_j(\mathbf{x})\, \mathrm{d}\mathbf{x} \tag{39}$$



Staggered Gaussian integration



FC-KL-expansion

FC-KL-expansion

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\hat{\lambda}_i}\, \hat{\varphi}_i(\mathbf{x})\, \xi_i \quad \text{with } \mathbf{x} \in \Omega \tag{40}$$

- **Note**: approx. eigenfunctions $\hat{\varphi}_i$ normalized w.r.t. Ω $\int_\Omega \hat{\varphi}_i(\mathbf{x})\hat{\varphi}_j(\mathbf{x})\,\mathrm{d}\mathbf{x} = \delta_{ij}$

Mean error variance

$$\overline{\varepsilon}_\sigma = 1 - \frac{1}{\sigma^2} \sum_{i=1}^{M} \hat{\lambda}_i \quad \text{with } \sigma = \sigma(\mathbf{x}) = \text{const.} \tag{41}$$

1D Example

Example

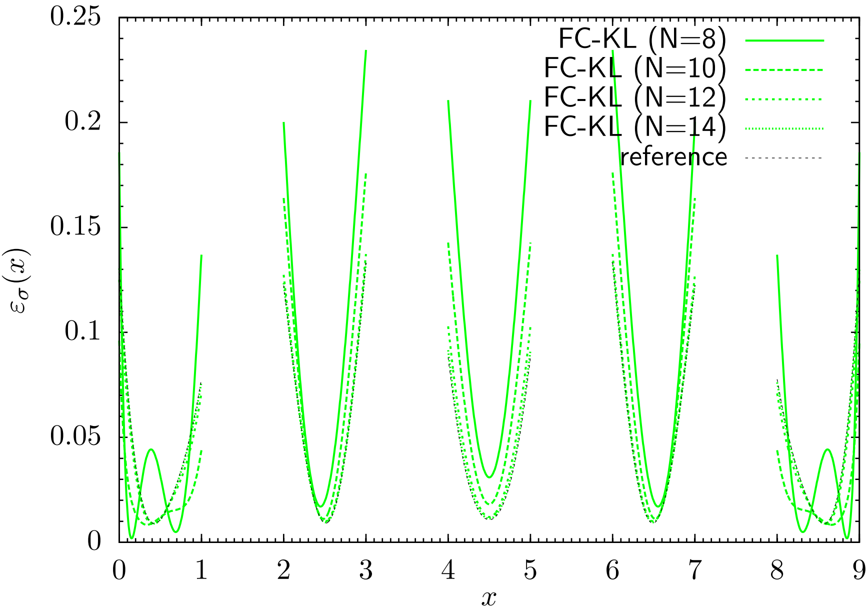
1D, straight domain

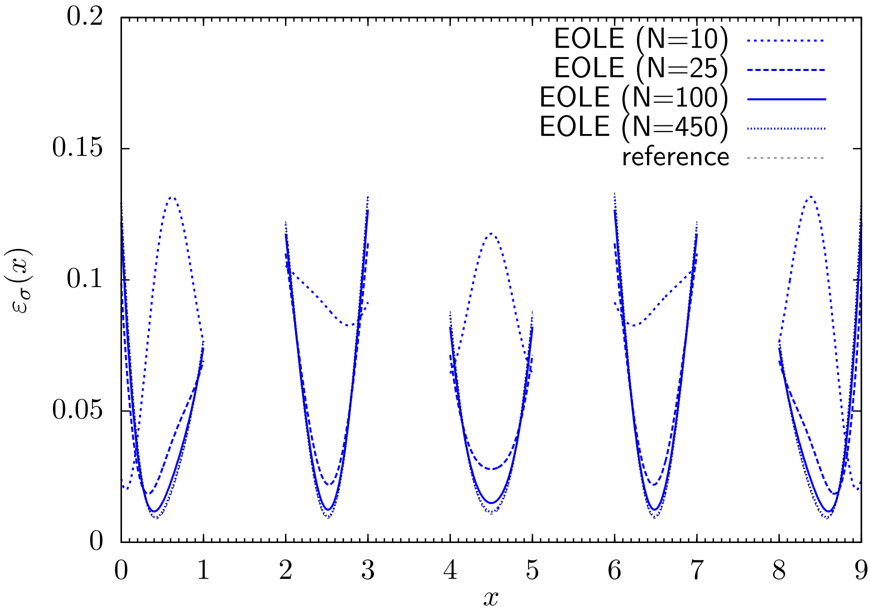
- length of domain: $l = 10$
- **geometry**: $\Omega = [0;1] \cup [2;3] \cup [4;5] \cup [6;7] \cup [8;9]$
- standard deviation: $\sigma = 1$
- correlation coefficient function: $\rho(x,x') = \exp\left(-\left(\frac{|x-x'|}{a}\right)^2\right)$
- correlation length: $a = 1$

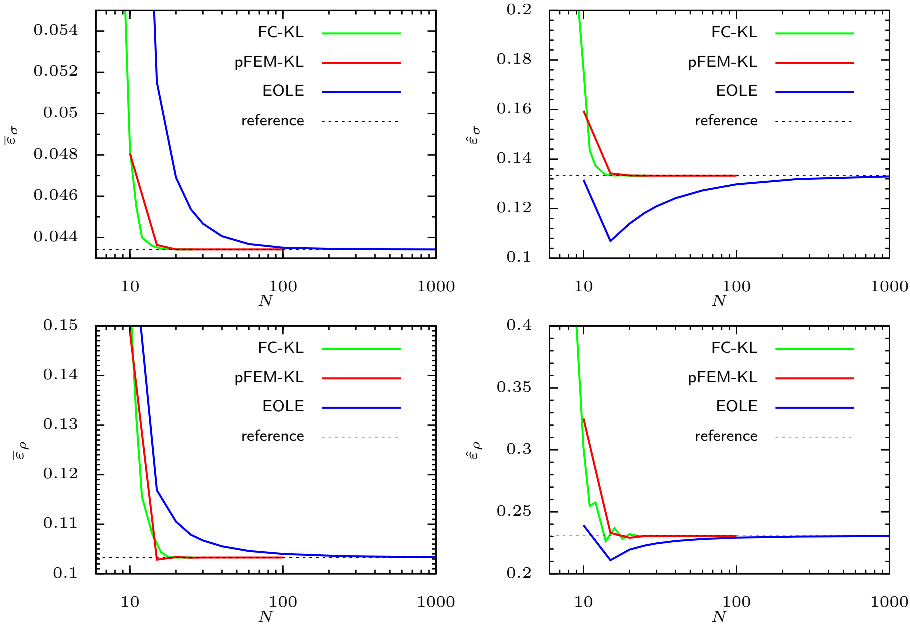
- FC-KL: just *one* element
- pFEM-KL: *five* elements

Parameter study

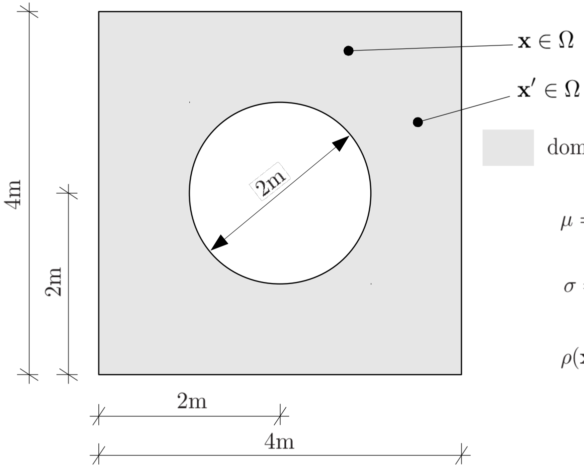
- **FIX** $M = 8$
- **MODIFY** N


FC-KL (N=8)
FC-KL (N=10)
FC-KL (N=12)
FC-KL (N=14)
reference
$\varepsilon_\sigma(x)$
x
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8
9


EOLE (N=10)
EOLE (N=25)
EOLE (N=100)
EOLE (N=450)
reference
$\varepsilon_\sigma(x)$
x
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8
9


FC-KL
pFEM-KL
EOLE
reference
$\bar{\varepsilon}_\sigma$
$\hat{\varepsilon}_\sigma$
$\bar{\varepsilon}_\rho$
$\hat{\varepsilon}_\rho$
N
10
100
1000
0.054
0.052
0.05
0.048
0.046
0.044
0.2
0.18
0.16
0.14
0.12
0.1
0.15
0.13
0.11
0.4
0.35
0.3
0.25

Example of a plate with a hole



domain Ω of the random field with

$$\mu = 30 \cdot 10^3 \frac{\text{MN}}{\text{m}^2}$$

$$\sigma = 6 \cdot 10^3 \frac{\text{MN}}{\text{m}^2}$$

$$\rho(\mathbf{x}, \mathbf{x}') = \exp\left(-\left(\frac{|\mathbf{x} - \mathbf{x}'|}{1.5}\right)^2\right)$$

$M = 20$

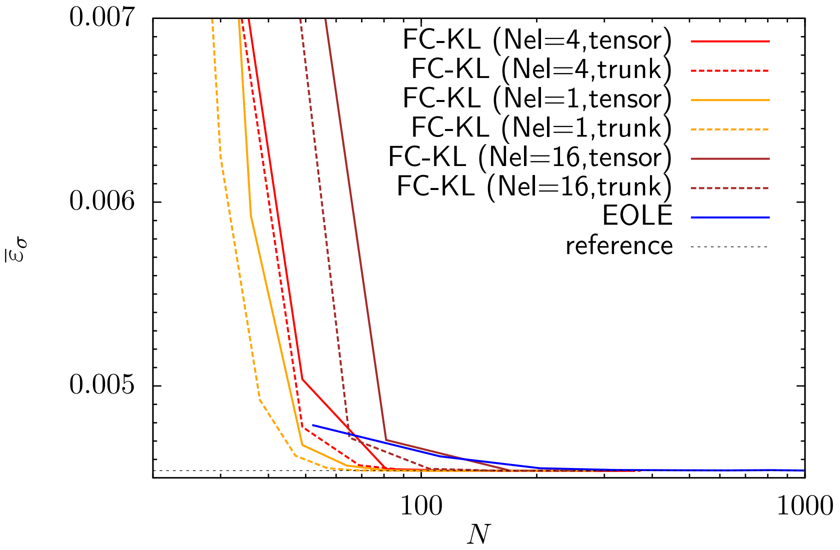

0.007
0.006
0.005
$\bar{\varepsilon}_\sigma$
100
1000
N
FC-KL (Nel=4,tensor)
FC-KL (Nel=4,trunk)
FC-KL (Nel=1,tensor)
FC-KL (Nel=1,trunk)
FC-KL (Nel=16,tensor)
FC-KL (Nel=16,trunk)
EOLE
reference

Table: size of the eigenvalue problem and time needed to achieve a mean error variance with a value of round $4.539 \cdot 10^{-3}$; two-dimensional example of a plate with a hole; $M = 20$

Method	N	time	RF info
EOLE	1040	$0.12s$	81 points per m^2
FC-KL ($N_{el} = 1$, tensor space)	100	$3.8s$	$p_{max} = 9$
FC-KL ($N_{el} = 1$, trunk space)	80	$4.9s$	$p_{max} = 11$
FC-KL ($N_{el} = 4$, tensor space)	169	$8.5s$	$p_{max} = 6$
FC-KL ($N_{el} = 4$, trunk space)	121	$8.1s$	$p_{max} = 7$
FC-KL ($N_{el} = 16$, tensor space)	≥ 289	$> 20s$	$p_{max} > 3$
FC-KL ($N_{el} = 16$, trunk space)	161	$22s$	$p_{max} = 4$

FEM post-processing of the random field

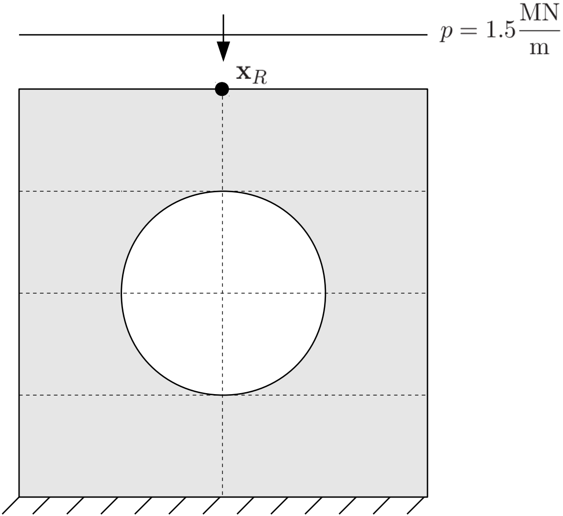


Table: time per finite cell run; rounded mean error variance of the random field approximation: $4.539 \cdot 10^{-3}$; two-dimensional example of a plate with a hole; $M = 20$

Method	N	time per FC run	p_{integ}	RF info
EOLE	1040	$65.30 \cdot 10^{-2}s$	6	81 p/m^2
FC-KL ($N_{el} = 4$, tensor)	169	$6.80 \cdot 10^{-2}s$	6	$p_{max} = 6$
FC-KL ($N_{el} = 4$, trunk)	121	$6.65 \cdot 10^{-2}s$	7	$p_{max} = 7$

Summary and Conclusion

FC-KL - Cons

- Computationally very **expensive to solve** *(especially in 3D)*
  - FC-KL: double integral over covariance function
  - EOLE: just $N \times N$ covariance function (and Lanczos methods)
- Quite **difficult to implement** *(compared to EOLE)*
  - pFEM
  - (double) integration of non-continuous non-smooth functions
- **Numerical stability** of eigenvalue problem

FC-KL - Pros

- **Fast convergence** against optimal representation
- Realization computationally **cheap to evaluate**
  - **Efficient assembly of FE stiffness matrices**
- **Simple mesh**