Discretization of Random Fields Based on the Karhunen-Loève Expansion Using the Finite Cell Method

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   - KL-expansion
2. FC-KL-exp.
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   - FC-KL-exp.
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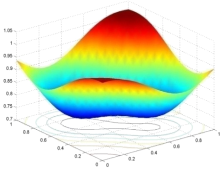
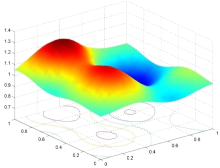
   - 2D-Example
4. Summary

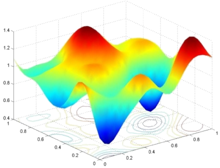
Motivation

Application of random fields - Examples:

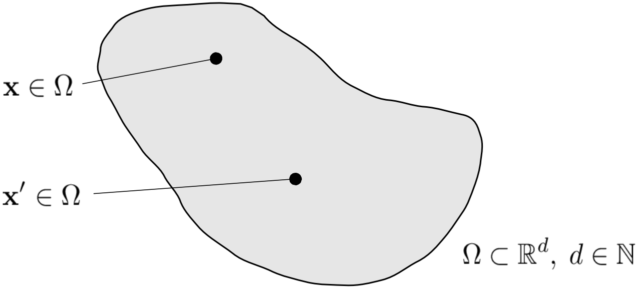
- soil properties in geotechnical engineering
- groundwater heights
- rainfall







Notation



- random field (RF): $H(\mathbf{x})$
- *Gaussian* random field - completely described by:
  - mean function $\mu(\mathbf{x})$
  - covariance function $\mathrm{Cov}(\mathbf{x}, \mathbf{x}') = \sigma(\mathbf{x}) \cdot \sigma(\mathbf{x}') \cdot \rho(\mathbf{x}, \mathbf{x}')$
    - $\sigma(\mathbf{x})$: standard deviation function
    - $\rho(\mathbf{x}, \mathbf{x}')$: correlation coefficient function

Random field discretization

Number of random variables (RVs) in a random field

- theoretically: *infinite* number of RVs (∞)
  - for each $\mathbf{x} \in \Omega$, $H(\mathbf{x})$ represents a RV
- discretized RF: *finite* number of RVs (M)

$$H(\mathbf{x}) \xrightarrow{\text{discretization}} \hat{H}(\mathbf{x}) \tag{1}$$

Categories of RF-discretization methods

- point discretization methods
- averaging discretization methods
- **series expansion methods**
  - Karhunen-Loève (KL) expansion
  - EOLE method

Karhunen-Loève expansion

KL-expansion

$$H(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\,\varphi_i(\mathbf{x})\,\xi_i \qquad (2)$$

- λ_i: eigenvalues of the covariance kernel
- φ_i: eigenfunctions of the covariance kernel
  - orthonormal: $\int_\Omega \varphi_i(\mathbf{x})\varphi_j(\mathbf{x})\,\mathrm{d}\mathbf{x} = \delta_{ij}$
- ξ_i: uncorrelated standard normal RVs
  - orthonormal: $\mathrm{E}[\xi_i\xi_j] = \delta_{ij}$

Integral eigenvalue problem

$$\int_{\mathbf{x}'\in\Omega} \varphi_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}' = \lambda_i\varphi_i(\mathbf{x}) \qquad (3)$$

Truncated KL-expansion

KL-expansion *(exact representation)*

$$H(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\,\varphi_i(\mathbf{x})\,\xi_i \tag{4}$$

Truncated KL-expansion *(approximation)*

$$\tilde{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\lambda_i}\,\varphi_i(\mathbf{x})\,\xi_i \tag{5}$$

- λ_i: M **largest** eigenvalues (in descending order)

Approximation of the KL-eigenfunctions

Integral eigenvalue problem (KL-expansion)

$$\int_{\mathbf{x}'\in\Omega} \varphi_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}' = \lambda_i\varphi_i(\mathbf{x}) \tag{6}$$

Approximation of the eigenfunctions

$$\hat{\varphi}_i(\mathbf{x}) = \sum_{n=1}^{N} d_n^i N_n(\mathbf{x}) = \mathbf{d}_i^T\,\mathbf{N}(\mathbf{x}) \tag{7}$$

- with $N_n(\mathbf{x}) \in L^2(\Omega)$

Minimization of the resulting error

Approximated integral eigenvalue problem

$$\int_{\mathbf{x}' \in \Omega} \hat{\varphi}_i(\mathbf{x}') \mathrm{Cov}(\mathbf{x}, \mathbf{x}') \, \mathrm{d}\mathbf{x}' - \hat{\lambda}_i \hat{\varphi}_i(\mathbf{x}) = \tilde{\varepsilon}_N^i(\mathbf{x}) \quad (8)$$

Minimization of the resulting error (Galerkin)

$$\int_{\Omega} \tilde{\varepsilon}_N^i(\mathbf{x}) N_k(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 0 \quad (9)$$

Matrix eigenvalue problem

Matrix eigenvalue problem

$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \tag{10}$$

where

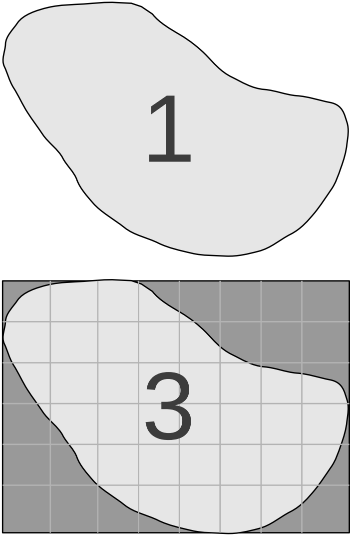
$$B_{kn} = \int\limits_{\mathbf{x}\in\Omega} N_k(\mathbf{x}) \int\limits_{\mathbf{x}'\in\Omega} N_n(\mathbf{x}')\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \tag{11}$$

$$M_{ij} = \int_{\mathbf{x}\in\Omega} N_i(\mathbf{x})N_j(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{12}$$

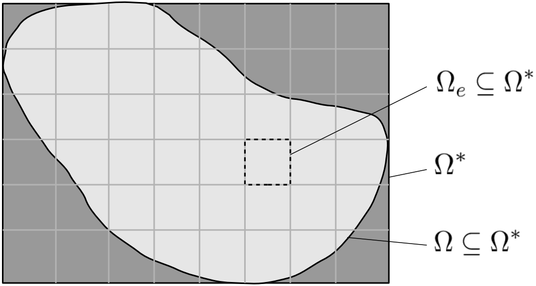
Approximated truncated KL-expansion

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\hat{\lambda}_i}\,\hat{\varphi}_i(\mathbf{x})\,\xi_i \tag{13}$$

Finite cell - basic idea



Finite cell - notation



- global shape functions: $N_i \in L^2(\Omega^*)$

$$\alpha(\mathbf{x}) = \begin{cases} 1 & \forall \mathbf{x} \in \Omega \\ 0 & \forall \mathbf{x} \in \Omega^* \setminus \Omega \end{cases} \tag{14}$$

Finite cell approach of the pFEM-KL-expansion

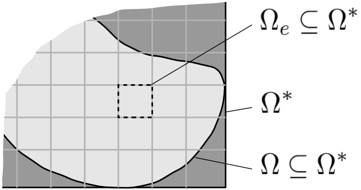
Matrix eigenvalue problem

$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \tag{15}$$

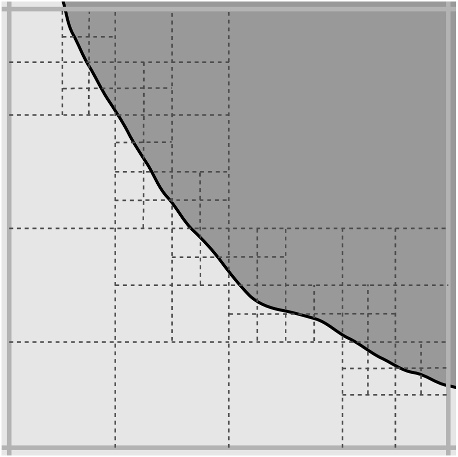
where

$$B_{kn} = \int_{\mathbf{x}\in\Omega^*} \alpha(\mathbf{x}) N_k(\mathbf{x}) \int_{\mathbf{x}'\in\Omega^*} \alpha(\mathbf{x}') N_n(\mathbf{x}') \mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \tag{16}$$

$$M_{ij} = \int_{\mathbf{x}\in\Omega^*} \alpha(\mathbf{x}) N_i(\mathbf{x}) N_j(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{17}$$



Staggered Gaussian integration



FC-KL-expansion

Matrix eigenvalue problem

$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \tag{18}$$

Approximation of the eigenfunctions

$$\hat{\varphi}_i(\mathbf{x}) = \sum_{n=1}^{N} d_n^i N_n(\mathbf{x}) = \mathbf{d}_i^T \mathbf{N}(\mathbf{x}) \tag{19}$$

- **Note**: approx. eigenfunctions $\hat{\varphi}_i$ normalized w.r.t. Ω $\int_\Omega \hat{\varphi}_i(\mathbf{x})\hat{\varphi}_j(\mathbf{x})\,\mathrm{d}\mathbf{x} = \delta_{ij}$

Type of the polynomial shape functions

Matrix eigenvalue problem to solve

$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \tag{20}$$

where

$$B_{kn} = \int_{\mathbf{x}\in\Omega^*} \alpha(\mathbf{x}) N_k(\mathbf{x}) \int_{\mathbf{x}'\in\Omega^*} \alpha(\mathbf{x}') N_n(\mathbf{x}') \mathrm{Cov}(\mathbf{x},\mathbf{x}') \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \tag{21}$$

$$M_{ij} = \int_{\mathbf{x}\in\Omega^*} \alpha(\mathbf{x}) N_i(\mathbf{x}) N_j(\mathbf{x}) \, \mathrm{d}\mathbf{x} \tag{22}$$

- Legendre polynomials
- hierarchic Legendre polynomials
- hierarchic Gegenbauer polynomials

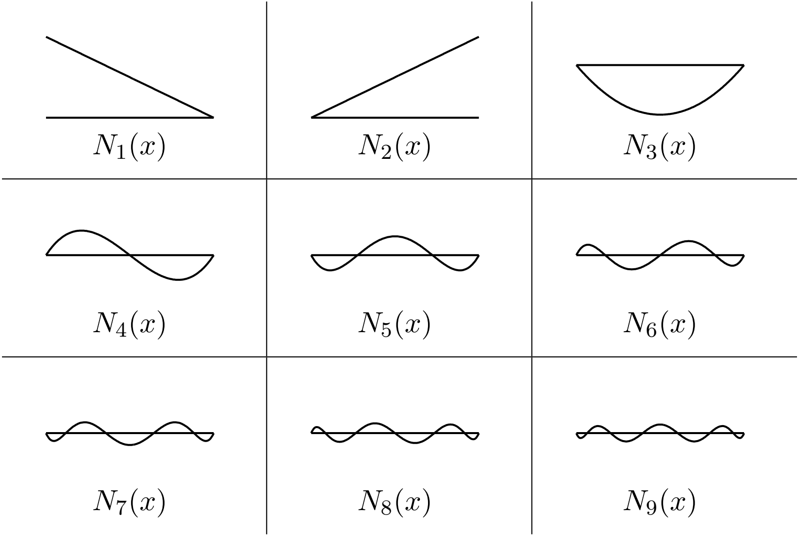
Legendre polynomials L_k (1D)

Orthogonality

$$\int_{-1}^{1} L_k(\xi) L_n(\xi)\, \mathrm{d}\xi = \frac{2}{2n+1}\, \delta_{kn} \tag{23}$$

- **Advantages**
  - perfect for a single element *(embedded domain)*
    - $\mathbf{Bd}_i = \hat{\lambda}_i \mathbf{Md}_i \quad \rightarrow \quad \mathbf{Bd}_i = \hat{\lambda}_i \mathbf{d}_i$
- **Disadvantages**
  - linking of different elements is difficult
  - refinement only through p_{max}
    - $p_{max} \geq 10$ *is numerically critical*

Hierarchic Legendre polynomials (1D)



Hierarchic Legendre polynomials L_k (1D)

Orthogonality

$$\int_{-1}^{1} \frac{\mathrm{d}N_i}{\mathrm{d}\xi}\frac{\mathrm{d}N_j}{\mathrm{d}\xi}\,\mathrm{d}\xi = \delta_{ij} \qquad i \geq 3 \wedge j \geq 1 \vee i \geq 1 \wedge j \geq 3 \tag{24}$$

- **Advantages**
  - hierarchic basis
  - coupling of elements straight forward
    - *nodal, face and edge modes*
- **Disadvantages**
  - orthogonality property does not favor mass matrix

Use a different hierarchic basis?

Requirements

- **hierarchic basis**
- $N_1(\xi) = \frac{1}{2}(1-\xi) \quad \wedge \quad N_2(\xi) = \frac{1}{2}(1+\xi)$
- $N_i(-1) = 0 \quad \wedge \quad N_i(1) = 0 \quad \forall \quad i > 2$
- $\int_{-1}^{1} N_i(\xi) N_j(\xi)\,\mathrm{d}\xi = \delta_{ij} \quad \forall \quad i,j > 2$

- $\int_{-1}^{1} N_i N_j = \int_{-1}^{1} (1-\xi^2)^2 \phi_{i-3}(\xi)\phi_{j-3}(\xi)\,\mathrm{d}\xi = \delta_{ij}\,\forall\, i,j > 2$
  - where $\deg(\phi_i) = i$
- Gegenbauer polynomials are orthogonal to weight function $(1-\xi^2)^{\alpha-\frac{1}{2}}$
- ϕ_i: use normalized Gegenbauer polynomials with $\alpha = 2.5$

Gegenbauer polynomials

Definition of the Gegenbauer polynomials for $\alpha = 2.5$

- $C_0^{2.5}(\xi) = 1$
- $C_1^{2.5}(\xi) = 5x$
- $C_i^{2.5}(\xi) = \frac{1}{n}\left[2\xi(n+1.5)C_{i-1}^{2.5}(\xi) - (n+3)C_{i-2}^{2.5}(\xi)\right]$

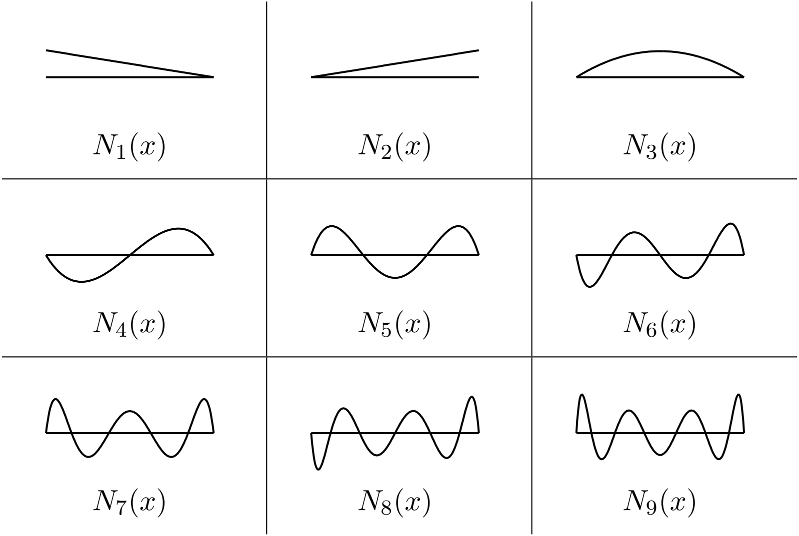
Normalized Gegenbauer polynomials

$$\phi_i(\xi) = \sqrt{\frac{n!(n+2.5)\left[\Gamma(2.5)\right]^2}{\pi 2^{-4}\Gamma(n+5)}} \cdot C_i^{2.5}(\xi) \tag{25}$$

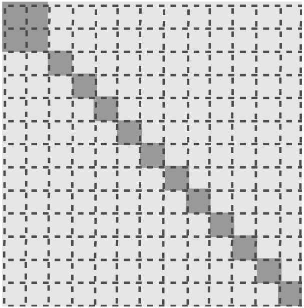
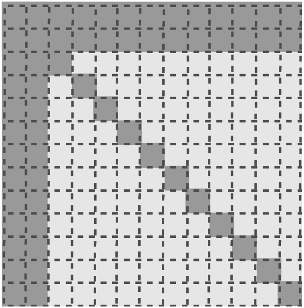
Hierarchic Gegenbauer polynomials

- $N_1(\xi) = \frac{1}{2}(1-\xi)$
- $N_2(\xi) = \frac{1}{2}(1+\xi)$
- $N_i(\xi) = (1-\xi^2)\cdot\phi_i(\xi) \quad \forall\, i > 2$

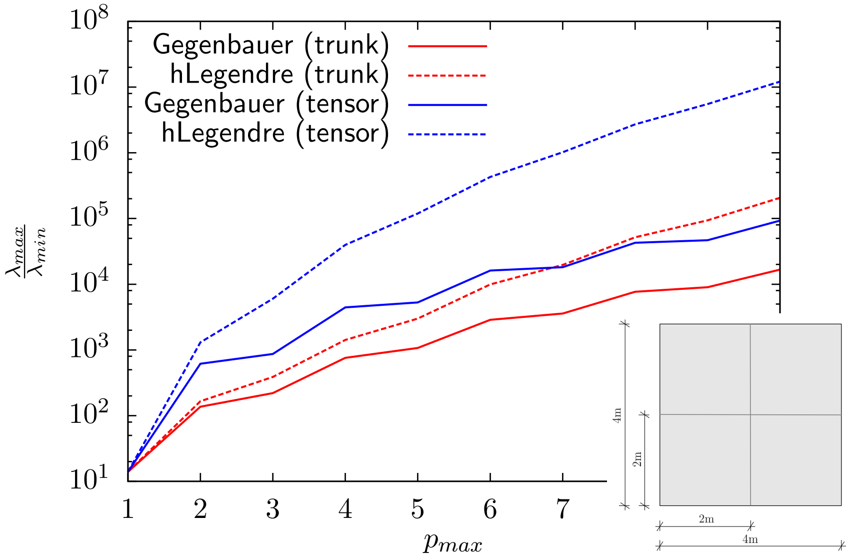
Hierarchic Gegenbauer polynomials (1D)



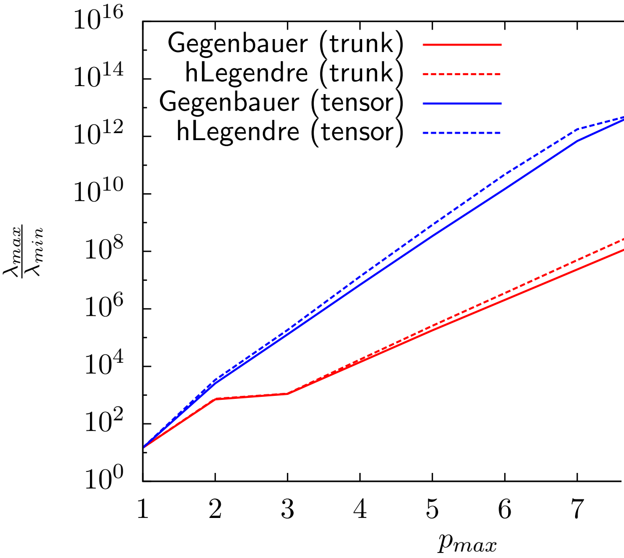
Comparison of the hierarchic bases

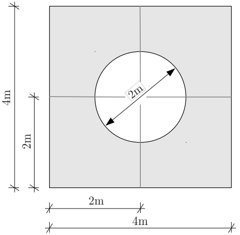
Stiffness matrix hierarchic **Legendere** polyn.	*Mass matrix* hierarchic **Gegenbauer** polyn.
	

Condition number of the mass matrix

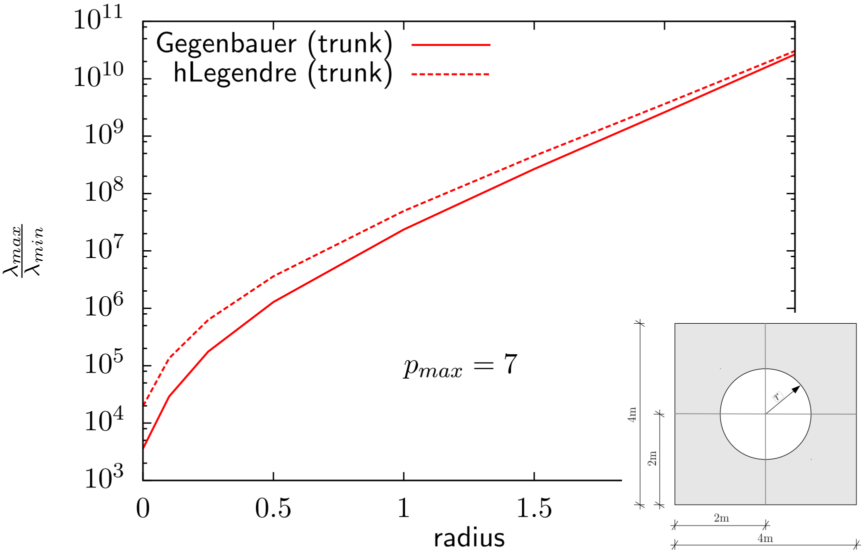


Condition number of the mass matrix





Condition number of the mass matrix



Error variance

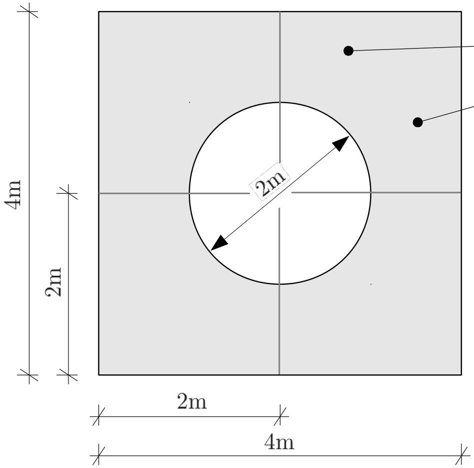
Error variance

$$\varepsilon_\sigma(\mathbf{x}) = \frac{\mathrm{Var}\left[H(\mathbf{x}) - \hat{H}(\mathbf{x})\right]}{\sigma^2(\mathbf{x})} \qquad (26)$$

Mean error variance

$$\varepsilon = \frac{\int_\Omega \varepsilon_\sigma(\mathbf{x})\,\mathrm{d}\mathbf{x}}{|\Omega|} \qquad (27)$$

Example of a plate with a hole



$\mathbf{x} \in \Omega$

$\mathbf{x}' \in \Omega$

domain Ω of the random field with

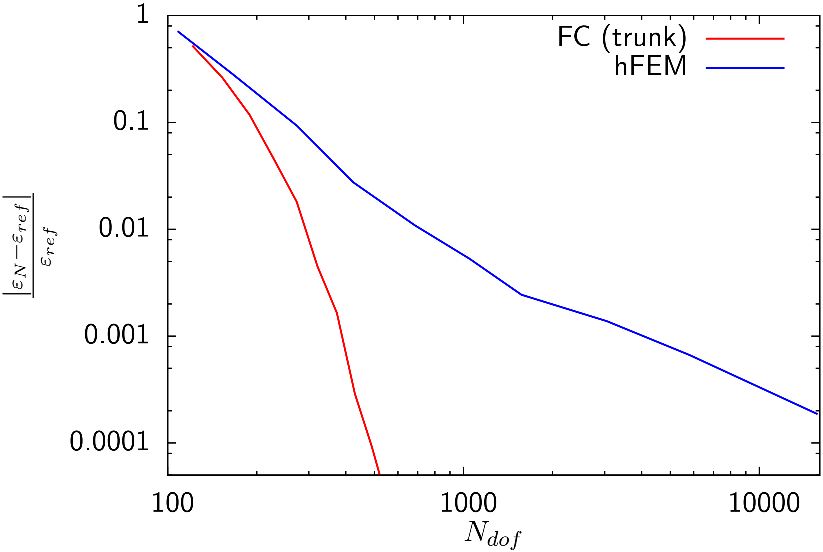
$$\mu = 30 \cdot 10^3 \frac{\mathrm{MN}}{\mathrm{m}^2}$$

$$\sigma = 6 \cdot 10^3 \frac{\mathrm{MN}}{\mathrm{m}^2}$$

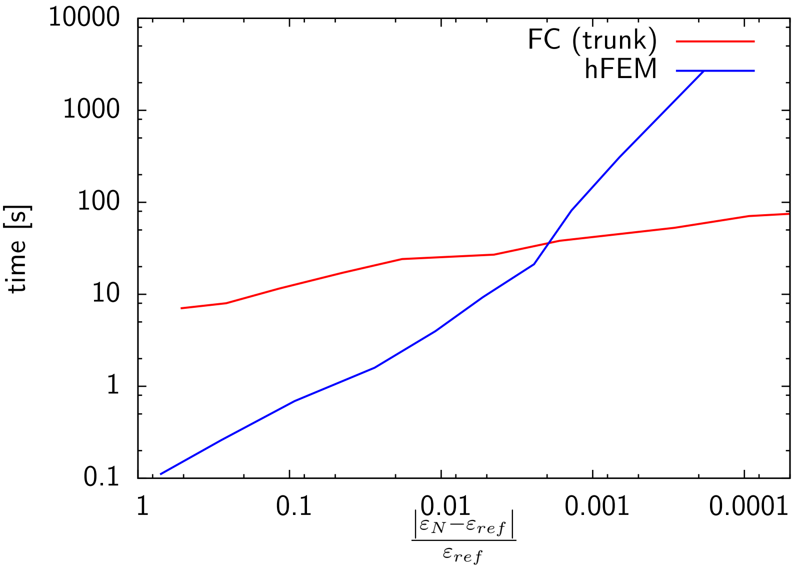
$$\rho(\mathbf{x}, \mathbf{x}') = \exp\left(-\left(\frac{|\mathbf{x} - \mathbf{x}'|}{0.3325}\right)^2\right)$$

$$\varepsilon_{ref} = 0.09978$$

$M = 100$: relative error



$M = 100$: time


FC (trunk)
hFEM
10000
1000
100
10
1
0.1
time [s]
1
0.1
0.01
0.001
0.0001
$\frac{|\varepsilon_N - \varepsilon_{ref}|}{\varepsilon_{ref}}$

Outlook - Spectral Stochastic FEM

Advantages

- implementation straightforward
  - ... only assembly of local stiffness matrices changes

Drawbacks *pFEM vs. hFEM*

- condensation of internal DOFs **NOT** allowed
  - ... condition number of *very* sparse matrix becomes worse

Outlook

- it is a question of the number of DOFs of the deterministic system
  - which one is 'better': *FC, pFEM, hFEM* ?

Summary and Conclusion

FC-KL - Cons

- Computationally very **expensive to solve** *(especially in 3D)*
  - FC-KL: double integral over covariance function
- Quite **difficult to implement**
  - pFEM
  - (double) integration of non-continuous non-smooth functions
- **Numerical stability** of eigenvalue problem

FC-KL - Pros

- **Fast convergence** against optimal representation
- Different requirements of FE- and RF-mesh can be easily taken into account
- **Simple mesh**