Quasi meshless discretization of random fields based on the Karhunen-Loève expansion

Wolfgang Betz

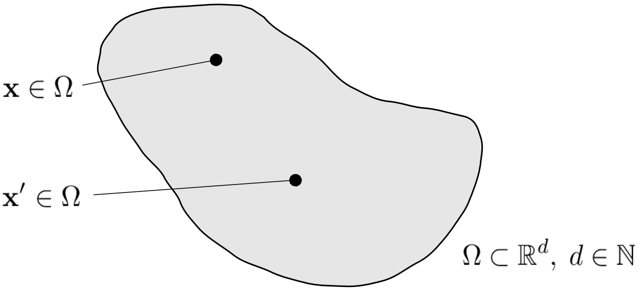
Reliability Seminar, UC Berkeley

2011-10-19

Abstract

The KL-expansion is a method for discretization of random fields, which is optimal among series expansion methods with respect to the mean square truncation error. The expansion is defined in terms of the eigenvalues and eigenfunctions of an integral eigenvalue problem with the autocovariance function as the integral kernel. Analytical solutions of this problem exist only for a few autocovariance functions and simple geometries. For the general case, the integral eigenvalue problem needs to be solved numerically, e.g. by application of the finite element (FE) method. However, FE methods require the construction of an FE-mesh, which may be a nontrivial task for domains with complex geometry. This talk introduces an alternative approach for the solution of the KL eigenvalue problem by application of the finite cell method. This method is based on embedding the actual domain in a larger domain defined by rectangular subdomains, called finite cells. The accurate representation of the geometry is dealt with through the integration procedure, which is part of the FE solution process. In addition, the method uses high order shape functions for the approximation of the solution inside the cells. This new approach is compared to the EOLE method, which is an alternative meshless method for the discretization of random fields in domains with arbitrary shape.

Notation



- random field (RF): $H(\mathbf{x})$
- *Gaussian* random field - completely described by:
  - mean function $\mu(\mathbf{x})$
  - covariance function $\mathrm{Cov}(\mathbf{x}, \mathbf{x}') = \sigma(\mathbf{x}) \cdot \sigma(\mathbf{x}') \cdot \rho(\mathbf{x}, \mathbf{x}')$
    - $\sigma(\mathbf{x})$: standard deviation function
    - $\rho(\mathbf{x}, \mathbf{x}')$: correlation coefficient function

Random field discretization

Number of random variables (RVs) in a random field

- theoretically: *infinite* number of RVs (∞)
  - for each $\mathbf{x} \in \Omega$, $H(\mathbf{x})$ represents a RV
- discretized RF: *finite* number of RVs (M)

$$H(\mathbf{x}) \xrightarrow{\text{discretization}} \hat{H}(\mathbf{x}) \tag{1}$$

Categories of RF-discretization methods

- point discretization methods
- averaging discretization methods
- **series expansion methods**
  - Karhunen-Loève (KL) expansion
  - EOLE method

Mean square error

Truncation error

$$\varepsilon_H(\mathbf{x}) = H(\mathbf{x}) - \hat{H}(\mathbf{x}) \tag{2}$$

- Assumption: $\mathrm{E}\left[H(\mathbf{x})\right] = \mathrm{E}\left[\hat{H}(\mathbf{x})\right] \Rightarrow \mathrm{E}\left[\hat{\varepsilon}_H(\mathbf{x})\right] = 0$

Mean square error

$$\mathrm{E}\left[\varepsilon_H^2(\mathbf{x})\right] = \mathrm{Var}\left[\varepsilon_H(\mathbf{x})\right] = \mathrm{E}\left[\left(H(\mathbf{x}) - \hat{H}(\mathbf{x})\right)^2\right] \tag{3}$$

- $\mathrm{E}\left[\varepsilon_H^2(\mathbf{x})\right] = \mathrm{Var}\left[H(\mathbf{x})\right] + \mathrm{Var}\left[\hat{H}(\mathbf{x})\right] - 2 \cdot \mathrm{Cov}\left[H(\mathbf{x}), \hat{H}(\mathbf{x})\right]$

Global mean square error

$$\bar{\varepsilon}_H^2(\mathbf{x}) = \int_\Omega \mathrm{E}\left[\varepsilon_H^2(\mathbf{x})\right]\, \mathrm{d}\mathbf{x} \tag{4}$$

Error variance

Error variance

$$\varepsilon_\sigma(\mathbf{x}) = \frac{\mathrm{Var}\left[H(\mathbf{x}) - \hat{H}(\mathbf{x})\right]}{\sigma^2(\mathbf{x})} \tag{5}$$

- $\varepsilon_\sigma(\mathbf{x}) = \mathrm{E}\left[\varepsilon_H^2(\mathbf{x})\right] / \sigma^2(\mathbf{x})$

Mean error variance

$$\bar{\varepsilon}_\sigma = \frac{\int_\Omega \varepsilon_\sigma(\mathbf{x})\,\mathrm{d}\mathbf{x}}{|\Omega|} \tag{6}$$

Supremum norm of the error variance

$$\hat{\varepsilon}_\sigma = \sup_{\mathbf{x}\in\Omega} \varepsilon_\sigma \tag{7}$$

Covariance error

Covariance error

$$\varepsilon_\rho(\mathbf{x}) = \frac{\int_{\mathbf{x}' \in \Omega} \left| \mathrm{Cov}\left(H(\mathbf{x}), H(\mathbf{x}')\right) - \mathrm{Cov}\left(\hat{H}(\mathbf{x}), \hat{H}(\mathbf{x}')\right) \right| \,\mathrm{d}\mathbf{x}'}{\int_{\mathbf{x}' \in \Omega} \left| \mathrm{Cov}\left(H(\mathbf{x}), H(\mathbf{x}')\right) \right| \,\mathrm{d}\mathbf{x}'} \quad (8)$$

Mean covariance error

$$\overline{\varepsilon}_\rho = \frac{\int_\Omega \varepsilon_\rho(\mathbf{x}) \,\mathrm{d}\mathbf{x}}{|\Omega|} \quad (9)$$

Supremum norm of the covariance error

$$\hat{\varepsilon}_\rho = \sup_{\mathbf{x} \in \Omega} \varepsilon_\rho \quad (10)$$

Karhunen-Loève expansion

KL-expansion

$$H(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, \varphi_i(\mathbf{x})\, \xi_i \tag{11}$$

- λ_i: eigenvalues of the covariance kernel
- φ_i: eigenfunctions of the covariance kernel
  - orthonormal: $\int_\Omega \varphi_i(\mathbf{x})\varphi_j(\mathbf{x})\,\mathrm{d}\mathbf{x} = \delta_{ij}$
- ξ_i: uncorrelated standard normal RVs
  - orthonormal: $\mathrm{E}[\xi_i \xi_j] = \delta_{ij}$

Integral eigenvalue problem

$$\int_{\mathbf{x}' \in \Omega} \varphi_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x}, \mathbf{x}')\,\mathrm{d}\mathbf{x}' = \lambda_i \varphi_i(\mathbf{x}) \tag{12}$$

Truncated KL-expansion

Truncated KL-expansion

$$\tilde{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\lambda_i}\, \varphi_i(\mathbf{x})\, \xi_i \qquad (13)$$

- λ_i: M **largest** eigenvalues (in descending order)

Variance

$$\tilde{\mathrm{Var}}(\mathbf{x}) = \sum_{i=1}^{M} \lambda_i\, \varphi_i^2(\mathbf{x}) \qquad (14)$$

Covariance

$$\tilde{\mathrm{Cov}}(\mathbf{x}, \mathbf{x}') = \sum_{i=1}^{M} \lambda_i\, \varphi_i(\mathbf{x})\, \varphi_i(\mathbf{x}') \qquad (15)$$

KL-truncation error

Mean square error

$$\mathrm{E}\left[\varepsilon_H^2(\mathbf{x})\right] = \sigma^2(\mathbf{x}) - \sum_{i=1}^{M} \lambda_i\, \varphi_i^2(\mathbf{x}) \qquad (16)$$

- **optimal** w.r.t. $\int_\Omega \mathrm{Var}\left[\varepsilon_H(\mathbf{x})\right] \mathrm{d}\mathbf{x} = \sum_{i=M+1}^{\infty} \lambda_i$

Error variance

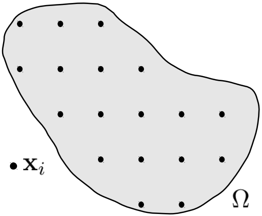
$$\varepsilon_\sigma(\mathbf{x}) = 1 - \frac{1}{\sigma^2(\mathbf{x})} \sum_{i=1}^{M} \lambda_i\, \varphi_i^2(\mathbf{x}) \qquad (17)$$

Mean error variance

$$\overline{\varepsilon}_\sigma = 1 - \frac{1}{\sigma^2} \sum_{i=1}^{M} \lambda_i \quad \text{with } \sigma = \sigma(\mathbf{x}) = \text{const.} \qquad (18)$$

EOLE method - basic idea

- model a RV χ_i at each $\mathbf{x}_i$
  - $(\boldsymbol{\Sigma}_{\chi\chi})_{nm} = \mathrm{Cov}(\mathbf{x}_n, \mathbf{x}_m)$
- solve eigenvalue problem:
  - $\boldsymbol{\Sigma}_{\chi\chi}\boldsymbol{\Phi}_i = \theta_i \boldsymbol{\Phi}_i$ (for M largest θ_i)
- $\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\theta_i} h_i(\mathbf{x})\xi_i$
  - $h_i(\mathbf{x}) = \boldsymbol{\Phi}_i^T \mathbf{b}(\mathbf{x})$
- find $\mathbf{b}^T(\mathbf{x})$ such that
  - minimize $\mathrm{Var}[\varepsilon_H(\mathbf{x})]$ subjected to $\mathrm{E}[\varepsilon_H] = 0$



EOLE-expansion

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \frac{\boldsymbol{\Phi}_i^T \boldsymbol{\Sigma}_{\chi\mathbf{x}}(\mathbf{x})}{\sqrt{\theta_i}} \xi_i \tag{19}$$

- with $(\boldsymbol{\Sigma}_{\chi\mathbf{x}}(\mathbf{x}))_j = \mathrm{Cov}(\mathbf{x}_j, \mathbf{x})$

EOLE method

EOLE-expansion

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \frac{\mathbf{\Phi}_i^T \mathbf{\Sigma}_{\chi\mathbf{x}}(\mathbf{x})}{\sqrt{\theta_i}} \, \xi_i \tag{20}$$

- $\left(\mathbf{\Sigma}_{\chi\mathbf{x}}(\mathbf{x})\right)_j = \mathrm{Cov}(\mathbf{x}_j, \mathbf{x})$
- solve $\mathbf{\Sigma}_{\chi\chi}\mathbf{\Phi}_i = \theta_i \mathbf{\Phi}_i$ with $\left(\mathbf{\Sigma}_{\chi\chi}\right)_{nm} = \mathrm{Cov}(\mathbf{x}_n, \mathbf{x}_m)$

Error variance

$$\varepsilon_\sigma(\mathbf{x}) = 1 - \frac{1}{\sigma^2(\mathbf{x})} \sum_{i=1}^{M} \frac{\left(\mathbf{\Phi}_i^T \mathbf{\Sigma}_{\chi\mathbf{x}}(\mathbf{x})\right)^2}{\theta_i} \tag{21}$$

- Note: geometry of Ω appears only *indirectly*

Approximation of the KL-eigenfunctions

Integral eigenvalue problem (KL-expansion)

$$\int_{\mathbf{x}'\in\Omega} \varphi_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}' = \lambda_i\varphi_i(\mathbf{x}) \tag{22}$$

Approximation of the eigenfunctions

$$\hat{\varphi}_i(\mathbf{x}) = \sum_{n=1}^{N} d_n^i N_n(\mathbf{x}) = \mathbf{d}_i^T\,\mathbf{N}(\mathbf{x}) \tag{23}$$

- with $N_n(\mathbf{x}) \in L^2(\Omega)$
- Remember - EOLE: $h_i(\mathbf{x}) = \mathbf{\Phi}_i^T\mathbf{b}(\mathbf{x})$; $\mathbf{\Phi}_i$ known *a priori*.

Minimization of the resulting error

Approximated integral eigenvalue problem

$$\int_{\mathbf{x}'\in\Omega} \hat{\varphi}_i(\mathbf{x}')\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}' - \hat{\lambda}_i\hat{\varphi}_i(\mathbf{x}) = \tilde{\varepsilon}_N^i(\mathbf{x}) \tag{24}$$

Minimization of the resulting error (Galerkin)

$$\int_\Omega \tilde{\varepsilon}_N^i(\mathbf{x})N_k(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \tag{25}$$

Matrix eigenvalue problem

Matrix eigenvalue problem

$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \tag{26}$$

where

$$B_{kn} = \int\limits_{\mathbf{x}\in\Omega} N_k(\mathbf{x}) \int\limits_{\mathbf{x}'\in\Omega} N_n(\mathbf{x}') \mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \tag{27}$$

$$M_{ij} = \int_{\mathbf{x}\in\Omega} N_i(\mathbf{x}) N_j(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{28}$$

Approximated truncated KL-expansion

Approximated truncated KL-expansion

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\hat{\lambda}_i}\, \hat{\varphi}_i(\mathbf{x})\, \xi_i \tag{29}$$

Error variance

$$\varepsilon_\sigma(\mathbf{x}) = 1 + \frac{\sum_{i=1}^{M} \hat{\lambda}_i\, \hat{\varphi}_i^2(\mathbf{x}) - 2\sum_{i=1}^{M} \hat{\varphi}_i(\mathbf{x}) \int_\Omega \mathrm{Cov}(\mathbf{x},\mathbf{x}')\hat{\varphi}_i(\mathbf{x}')\,\mathrm{d}\mathbf{x}'}{\sigma^2(\mathbf{x})} \tag{30}$$

Mean error variance

$$\overline{\varepsilon}_\sigma = 1 - \frac{1}{\sigma^2}\sum_{i=1}^{M} \hat{\lambda}_i \quad \text{with } \sigma = \sigma(\mathbf{x}) = \text{const.} \tag{31}$$

Optimality of KL, EOLE, approximated-KL

KL, *EOLE* and *approximated-KL* are based on a conditioned minimization of $\mathrm{Var}[\varepsilon_H(\mathbf{x})]$.

Karhunen-Loève

minimization *(globally)* w.r.t. any orthogonal basis of $L^2(\Omega)$

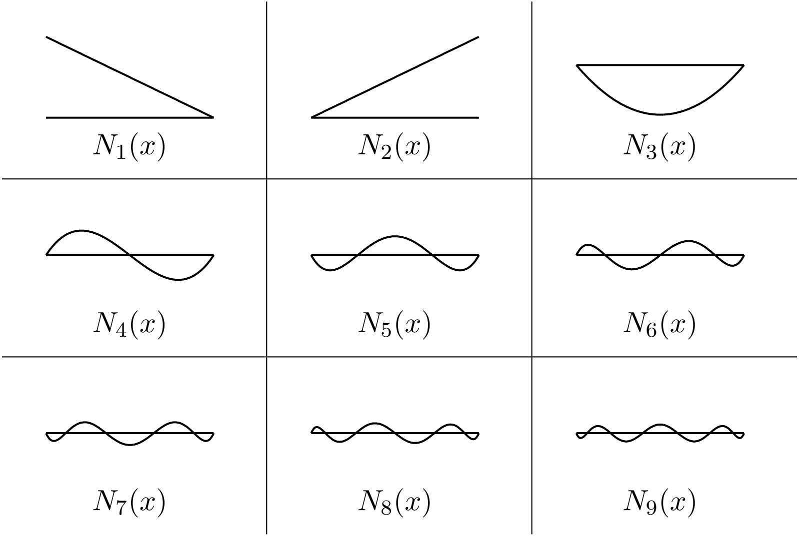
approximated-KL

minimization *(globally)* w.r.t. any orthogonal basis spanning the subspace of $L^2(\Omega)$ defined by $\{N_i(\mathbf{x})\}_{i=1}^{N}$

EOLE

minimization *(point-wise)* w.r.t. the chosen points $\{\mathbf{x}_i\}_{i=1}^{N}$

Hierarchic Legendre polynomials (1D)



1D Example

Example

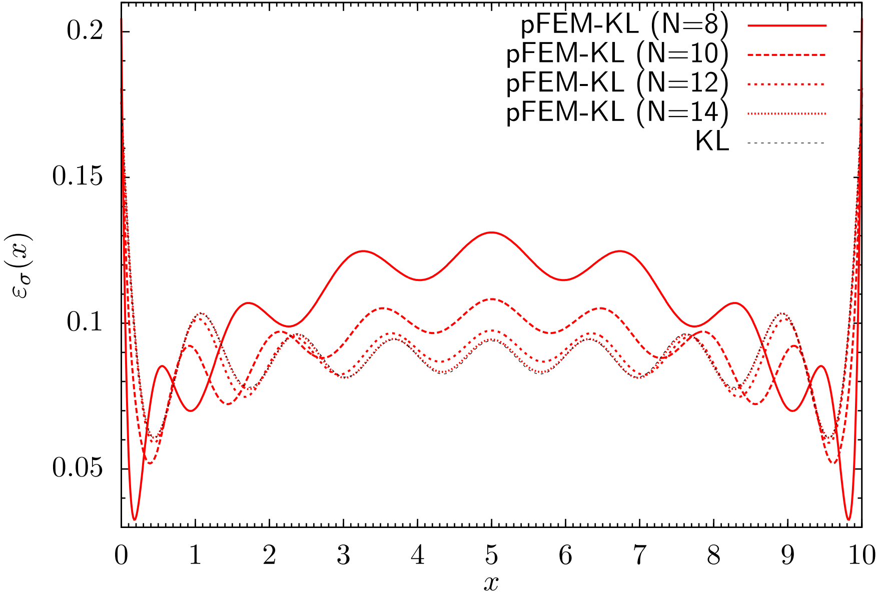
1D, straight domain

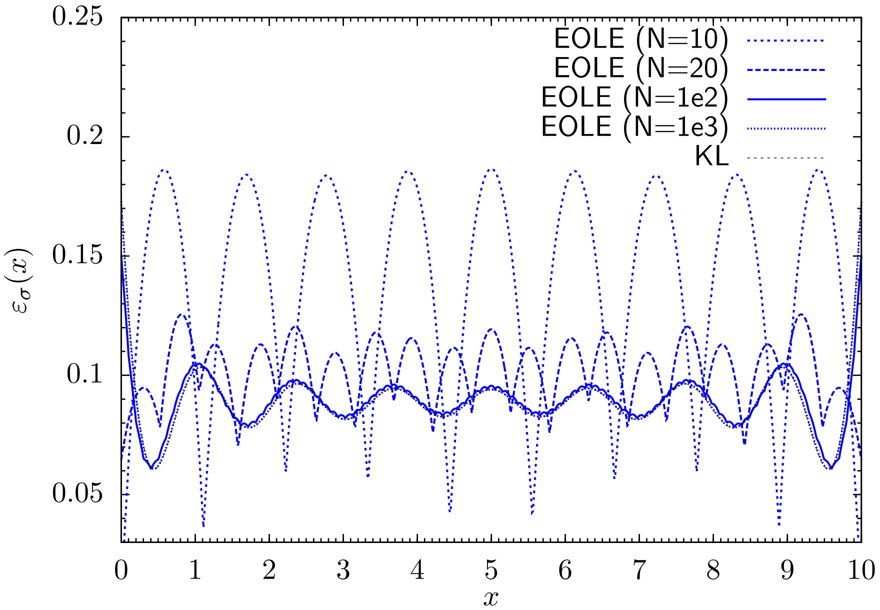
- length of domain: $l = 10$
- standard deviation: $\sigma = 1$
- correlation coefficient function: $\rho(x, x') = \exp\left(-\frac{|x-x'|}{a}\right)$
- correlation length: $a = 3$

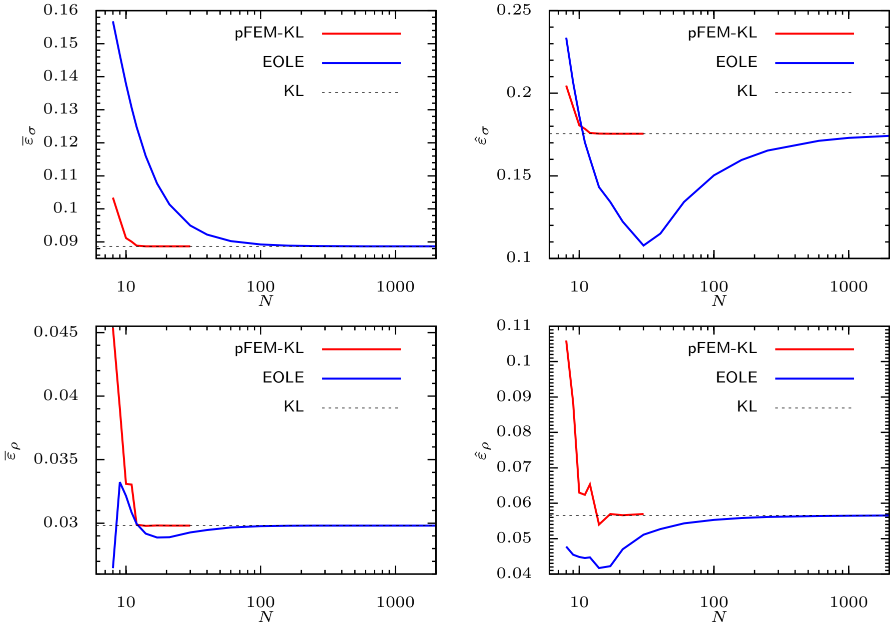
- pFEM-KL: just *one* element

Parameter study

- **FIX** $M = 8$
- **MODIFY** N


pFEM-KL (N=8)
pFEM-KL (N=10)
pFEM-KL (N=12)
pFEM-KL (N=14)
KL
0.2
0.15
0.1
0.05
$\varepsilon_\sigma(x)$
0 1 2 3 4 5 6 7 8 9 10
x


0.25
0.2
0.15
0.1
0.05
$\varepsilon_\sigma(x)$
0
1
2
3
4
5
6
7
8
9
10
x
EOLE (N=10)
EOLE (N=20)
EOLE (N=1e2)
EOLE (N=1e3)
KL


0.16
0.15
0.14
0.13
0.12
0.11
0.1
0.09
$\bar{\varepsilon}_\sigma$
pFEM-KL
EOLE
KL
10
100
1000
N
0.25
0.2
0.15
0.1
$\hat{\varepsilon}_\sigma$
pFEM-KL
EOLE
KL
10
100
1000
N
0.045
0.04
0.035
0.03
$\bar{\varepsilon}_\rho$
pFEM-KL
EOLE
KL
10
100
1000
N
0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
$\hat{\varepsilon}_\rho$
pFEM-KL
EOLE
KL
10
100
1000
N

1D Example

Example

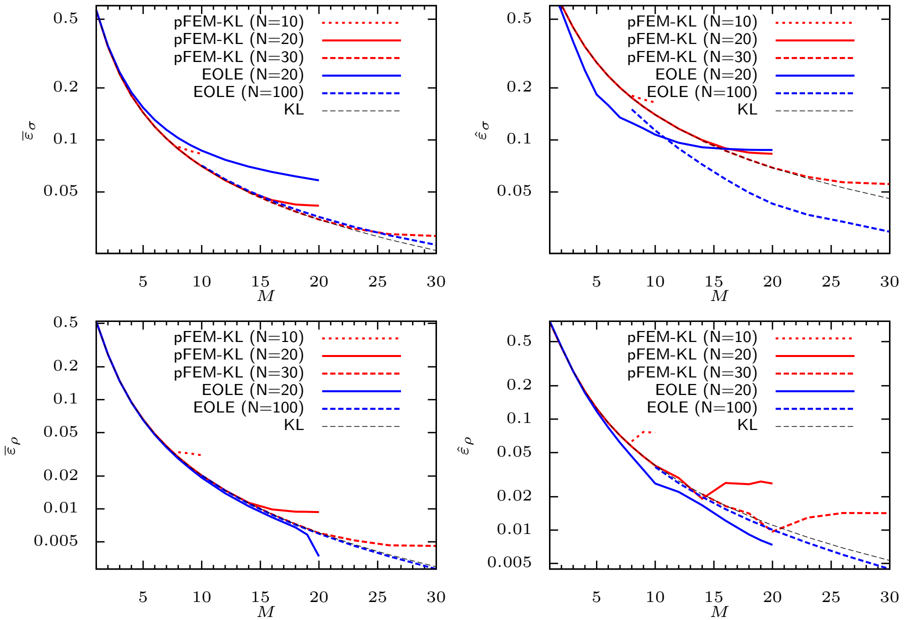
1D, straight domain

- length of domain: $l = 10$
- standard deviation: $\sigma = 1$
- correlation coefficient function: $\rho(x, x') = \exp\left(-\frac{|x-x'|}{a}\right)$
- correlation length: $a = 3$

- pFEM-KL: just *one* element

Parameter study

- **FIX** N
- **MODIFY** M


pFEM-KL (N=10)
pFEM-KL (N=20)
pFEM-KL (N=30)
EOLE (N=20)
EOLE (N=100)
KL
$\bar{\varepsilon}_\sigma$
$\hat{\varepsilon}_\sigma$
$\bar{\varepsilon}_\rho$
$\hat{\varepsilon}_\rho$
M
0.5
0.2
0.1
0.05
0.02
0.01
0.005
5
10
15
20
25
30

1D Example

Example

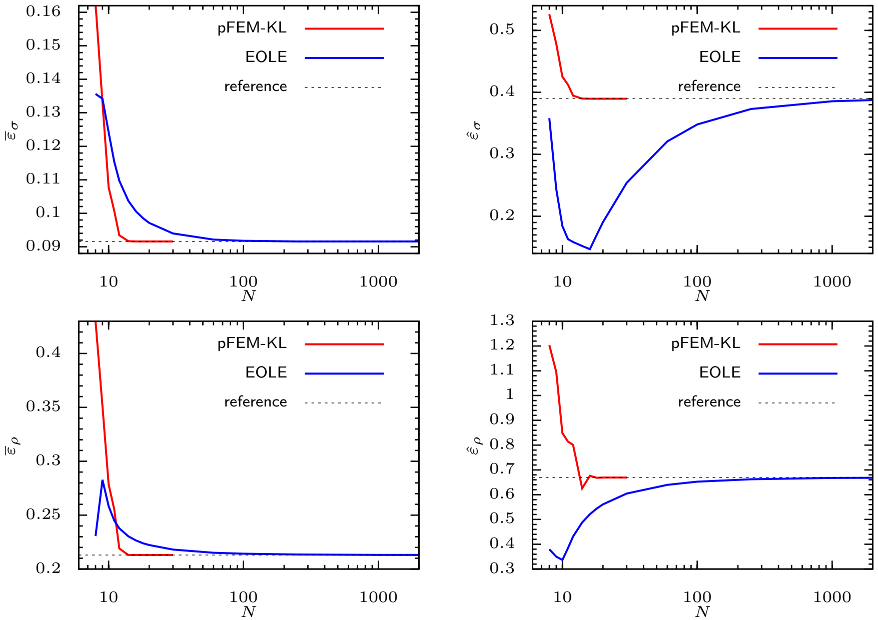
1D, straight domain

- length of domain: $l = 10$
- standard deviation: $\sigma = 1$
- **correlation coefficient function**:
  $\rho(x, x') = \exp\left(-\left(\frac{x-x'}{a}\right)^2\right)$
- **correlation length**: $a = 1$

- pFEM-KL: just *one* element

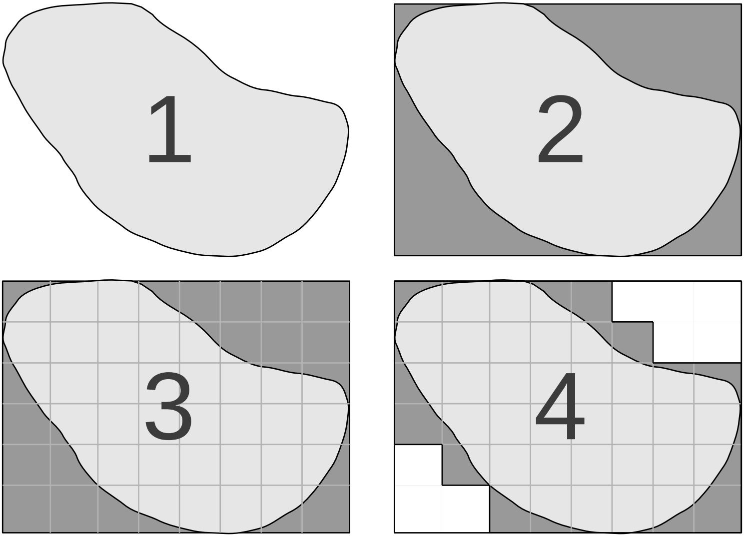
Parameter study

- **FIX** $M = 8$
- **MODIFY** N

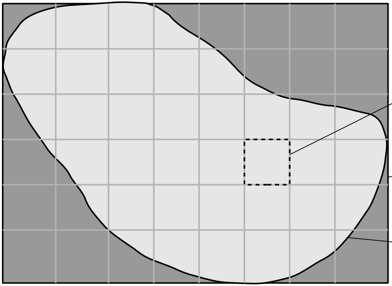


0.16
0.15
0.14
0.13
0.12
0.11
0.1
0.09
$\bar{\varepsilon}_\sigma$
pFEM-KL
EOLE
reference
10
100
1000
N
0.5
0.4
0.3
0.2
$\hat{\varepsilon}_\sigma$
0.4
0.35
0.3
0.25
0.2
$\bar{\varepsilon}_\rho$
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
$\hat{\varepsilon}_\rho$

Finite cell - basic idea



Finite cell - notation



- global shape functions: $N_i \in L^2(\Omega^*)$

$$\alpha(\mathbf{x}) = \begin{cases} 1 & \forall \mathbf{x} \in \Omega \\ 0 & \forall \mathbf{x} \in \Omega^* \setminus \Omega \end{cases} \tag{32}$$

Finite cell approach of the pFEM-KL-expansion

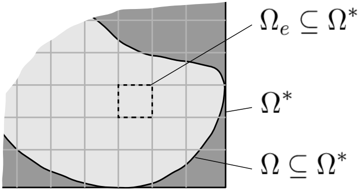
Matrix eigenvalue problem

$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \quad (33)$$

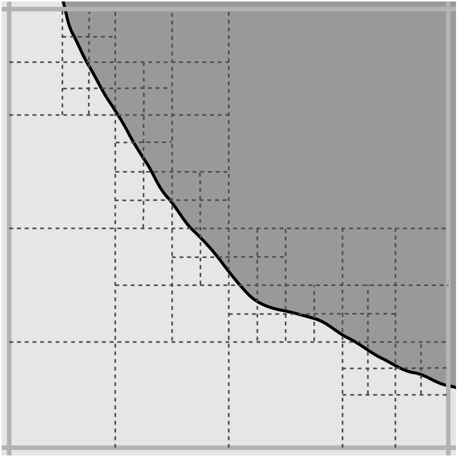
where

$$B_{kn} = \int_{\mathbf{x}\in\Omega^*} \alpha(\mathbf{x}) N_k(\mathbf{x}) \int_{\mathbf{x}'\in\Omega^*} \alpha(\mathbf{x}') N_n(\mathbf{x}') \mathrm{Cov}(\mathbf{x},\mathbf{x}')\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} \quad (34)$$

$$M_{ij} = \int_{\mathbf{x}\in\Omega^*} \alpha(\mathbf{x}) N_i(\mathbf{x}) N_j(\mathbf{x})\, \mathrm{d}\mathbf{x} \quad (35)$$



Staggered Gaussian integration



FC-KL-expansion

FC-KL-expansion

$$\hat{H}(\mathbf{x}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\hat{\lambda}_i}\, \hat{\varphi}_i(\mathbf{x})\, \xi_i \quad \text{with } \mathbf{x} \in \Omega \tag{36}$$

- **Note**: approx. eigenfunctions $\hat{\varphi}_i$ normalized w.r.t. Ω $\int_\Omega \hat{\varphi}_i(\mathbf{x})\hat{\varphi}_j(\mathbf{x})\,\mathrm{d}\mathbf{x} = \delta_{ij}$

Mean error variance

$$\overline{\varepsilon}_\sigma = 1 - \frac{1}{\sigma^2} \sum_{i=1}^{M} \hat{\lambda}_i \quad \text{with } \sigma = \sigma(\mathbf{x}) = \text{const.} \tag{37}$$

1D Example

Example

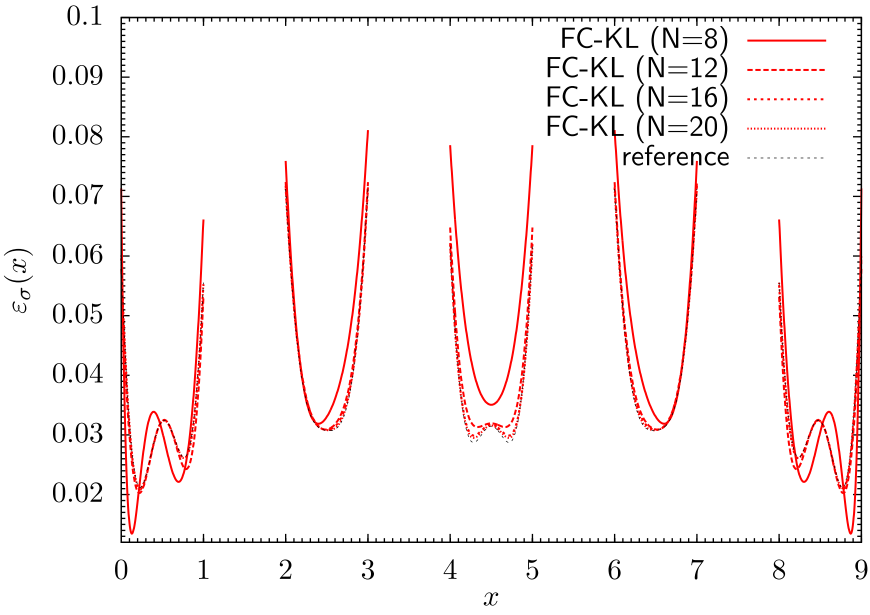
1D, straight domain

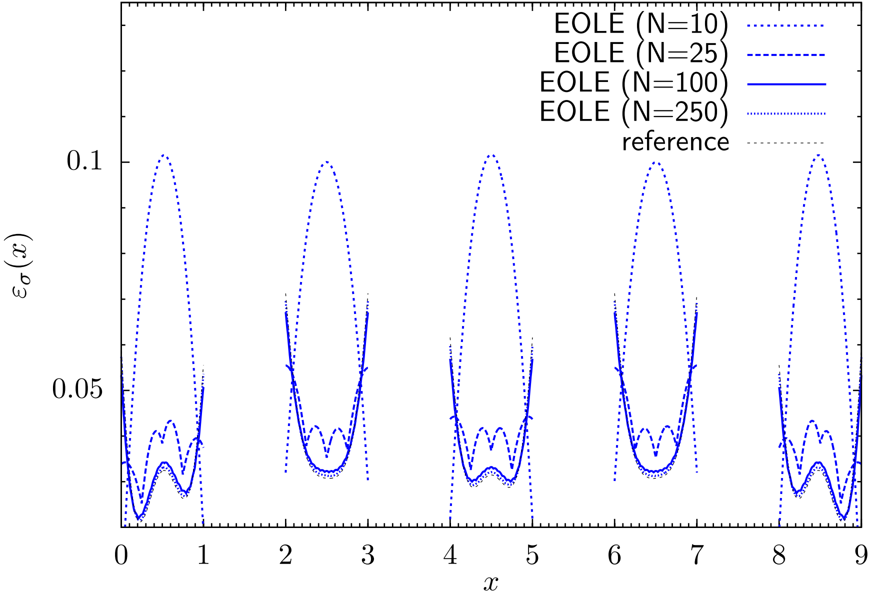
- length of domain: $l = 10$
- **geometry**: $\Omega = [0;1] \cup [2;3] \cup [4;5] \cup [6;7] \cup [8;9]$
- standard deviation: $\sigma = 1$
- correlation coefficient function: $\rho(x, x') = \exp\left(-\frac{|x-x'|}{a}\right)$
- correlation length: $a = 5$

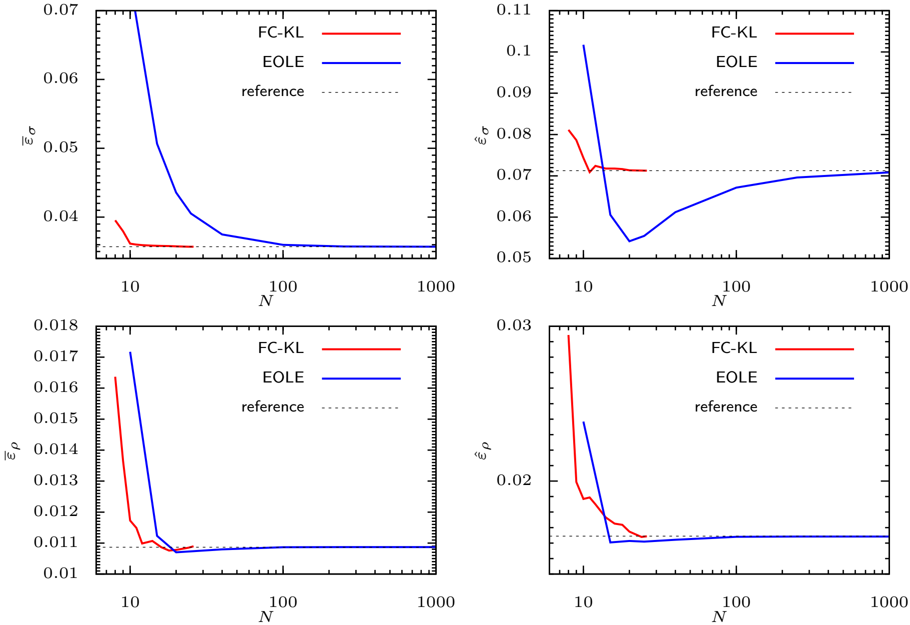
- FC-KL: just *one* element

Parameter study

- **FIX** $M = 8$
- **MODIFY** N


FC-KL (N=8)
FC-KL (N=12)
FC-KL (N=16)
FC-KL (N=20)
reference
$\varepsilon_\sigma(x)$
x
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0
1
2
3
4
5
6
7
8
9


EOLE (N=10)
EOLE (N=25)
EOLE (N=100)
EOLE (N=250)
reference
0.1
0.05
$\varepsilon_\sigma(x)$
0 1 2 3 4 5 6 7 8 9
x


0.07
0.06
0.05
0.04
$\bar{\varepsilon}_\sigma$
FC-KL
EOLE
reference
10
100
1000
N
0.11
0.1
0.09
0.08
0.07
0.06
0.05
$\hat{\varepsilon}_\sigma$
FC-KL
EOLE
reference
10
100
1000
N
0.018
0.017
0.016
0.015
0.014
0.013
0.012
0.011
0.01
$\bar{\varepsilon}_\rho$
FC-KL
EOLE
reference
10
100
1000
N
0.03
0.02
$\hat{\varepsilon}_\rho$
FC-KL
EOLE
reference
10
100
1000
N

1D Example

Example

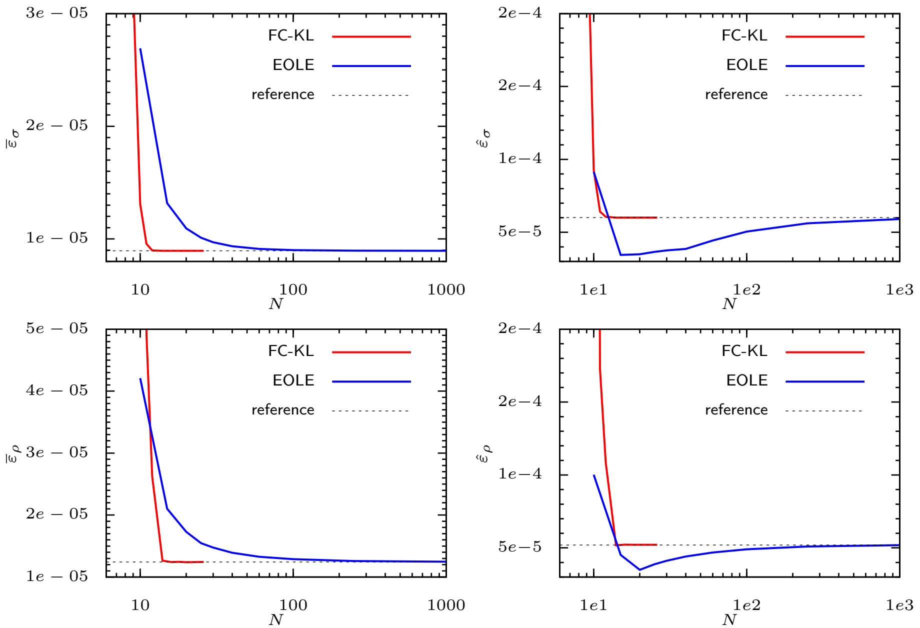
1D, straight domain

- length of domain: $l = 10$
- **geometry**: $\Omega = [0;1] \cup [2;3] \cup [4;5] \cup [6;7] \cup [8;9]$
- standard deviation: $\sigma = 1$
- correlation coefficient function: $\rho(x, x') = \exp\left(-\left(\frac{x-x'}{a}\right)^2\right)$
- correlation length: $a = 3$

- FC-KL: just *one* element

Parameter study

- **FIX** $M = 8$
- **MODIFY** N


3e − 05
2e − 05
1e − 05
$\bar{\varepsilon}_\sigma$
FC-KL
EOLE
reference
10
100
1000
N
2e−4
2e−4
1e−4
5e−5
$\hat{\varepsilon}_\sigma$
1e1
1e2
1e3
5e − 05
4e − 05
3e − 05
2e − 05
1e − 05
$\bar{\varepsilon}_\rho$
$\hat{\varepsilon}_\rho$

1D Example

Example

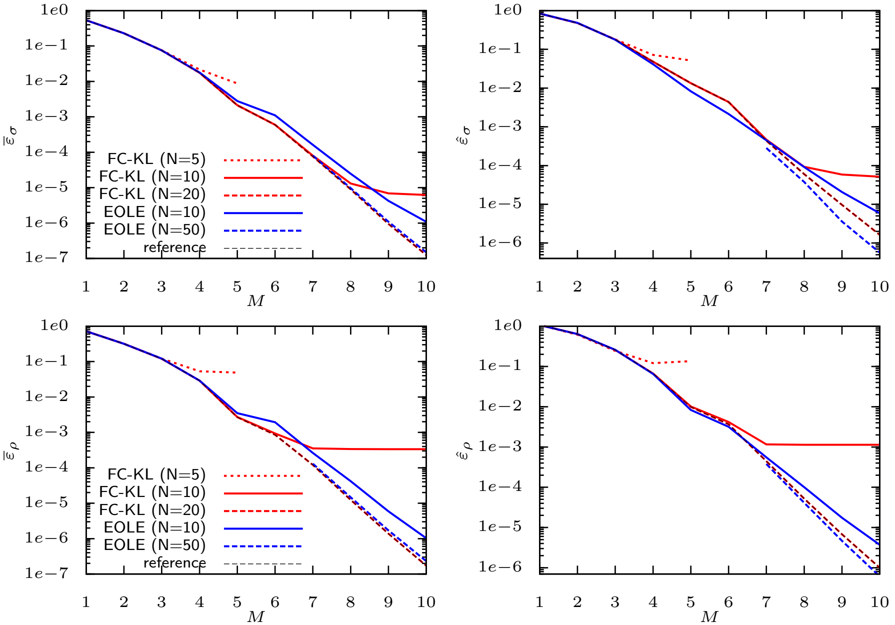
1D, straight domain

- length of domain: $l = 10$
- **geometry**: $\Omega = [0;1] \cup [2;3] \cup [4;5] \cup [6;7] \cup [8;9]$
- standard deviation: $\sigma = 1$
- correlation coefficient function: $\rho(x,x') = \exp\left(-\left(\frac{x-x'}{a}\right)^2\right)$
- correlation length: $a = 3$

- FC-KL: just *one* element

Parameter study

- **FIX** N
- **MODIFY** M


1e0
1e−1
1e−2
1e−3
1e−4
1e−5
1e−6
1e−7
$\bar{\varepsilon}_\sigma$
$\hat{\varepsilon}_\sigma$
$\bar{\varepsilon}_\rho$
$\hat{\varepsilon}_\rho$
1 2 3 4 5 6 7 8 9 10
M
FC-KL (N=5)
FC-KL (N=10)
FC-KL (N=20)
EOLE (N=10)
EOLE (N=50)
reference

Summary and Conclusion

FC-KL - Pros and Cons

- **Fast convergence** against optimal representation
- Computationally very **expensive to solve**
  - FC-KL: double integral over covariance function
  - EOLE: just $N \times N$ covariance function (and Lanczos methods)
- Realization computationally **cheap to evaluate**
- **Simple mesh**
- Quite **difficult to implement** (compared to EOLE)
  - pFEM
  - (double) integration of non-continuous non-smooth functions

Outlook

- solve higher-dimensional (2D and 3D) problems
- investigate numerical stability of FC-approach