14.07.2011

BGCE Research Day

Polynomial Chaos Approximations of Random Variables

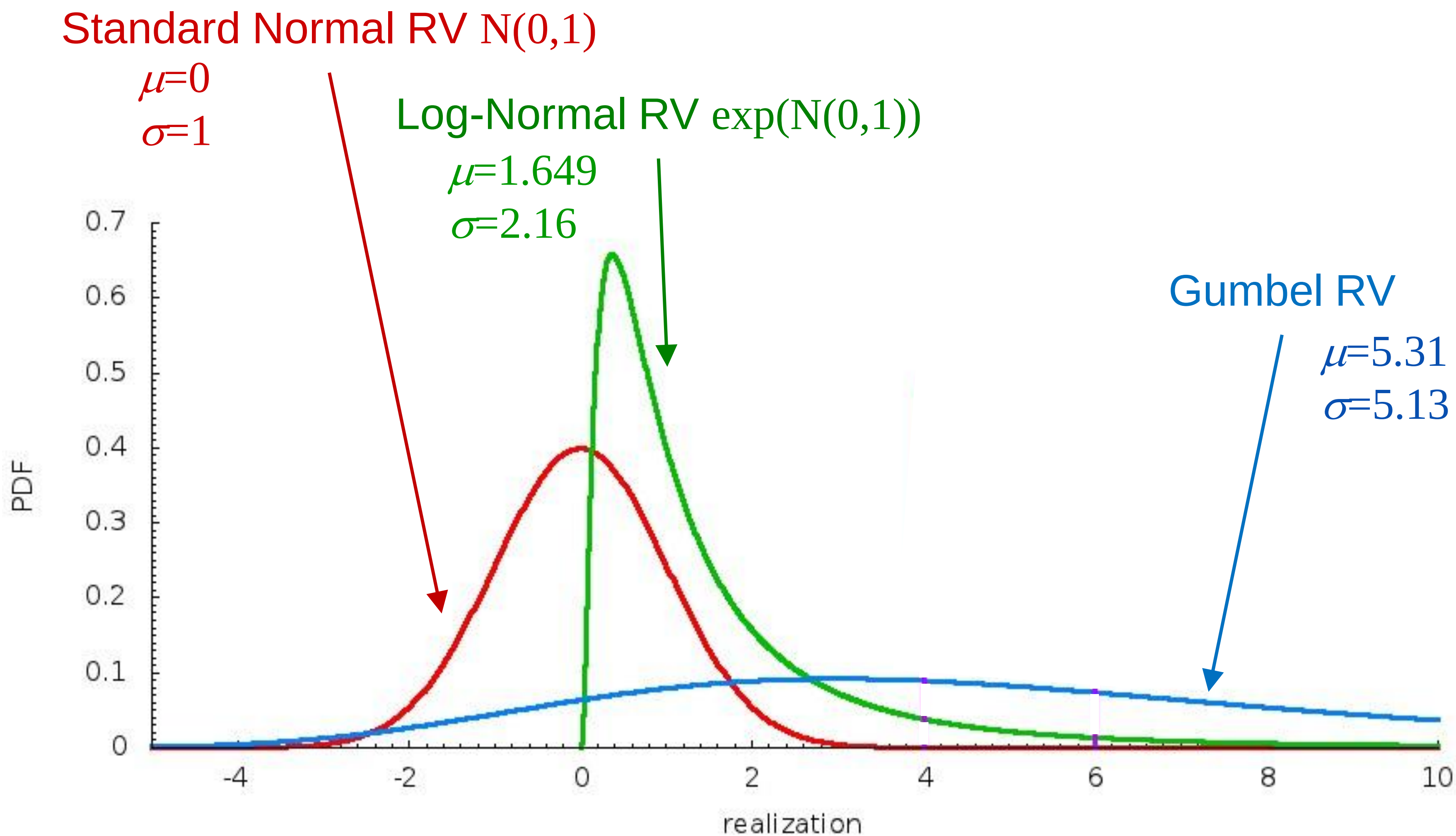
non intrusive spectral methods

Wolfang Betz

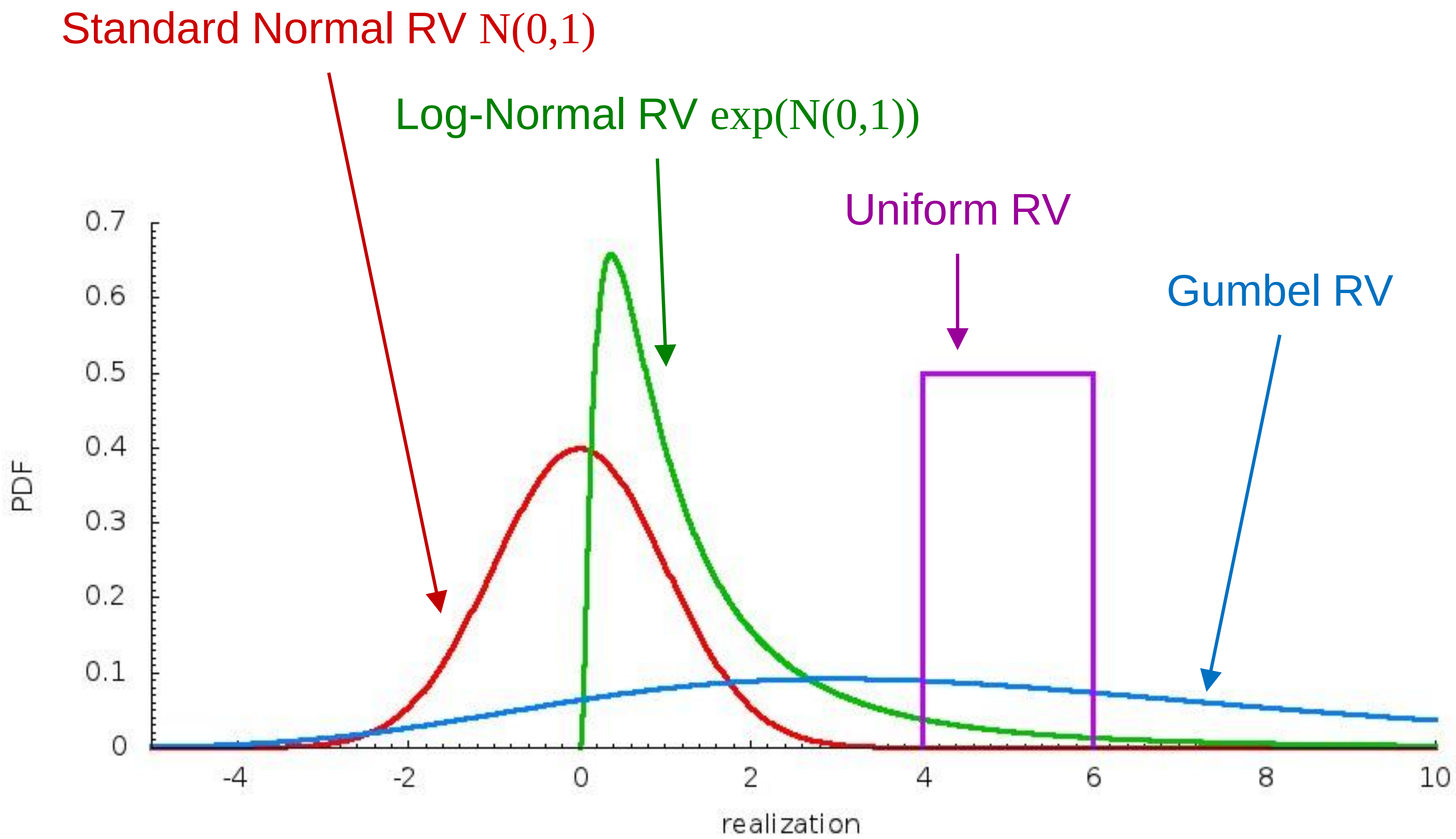
Outline

- Random variables
- Stochastic models
- Polynomial Chaos (approximation)
- Quality of the approximation
  - 1D
  - higher dimensions

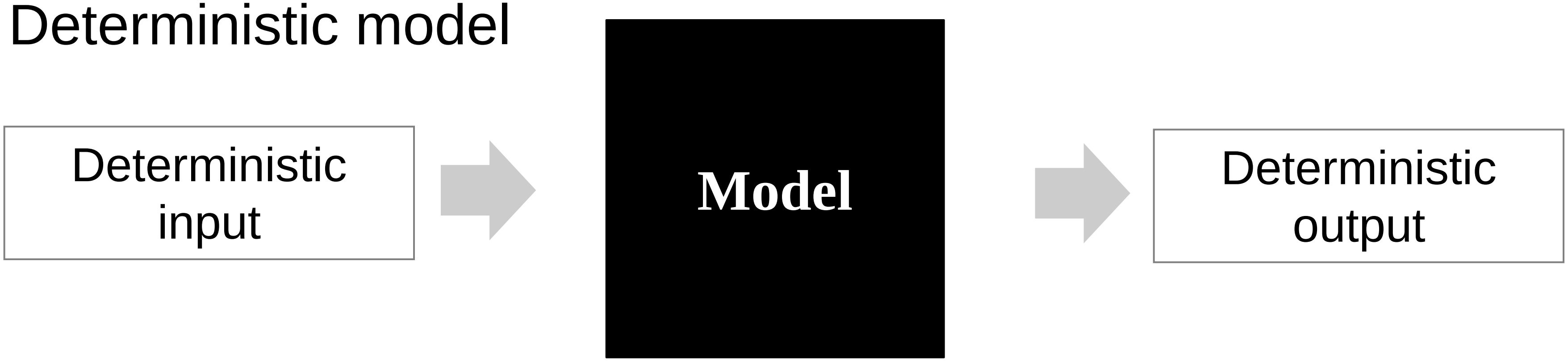
Distributions of random variables



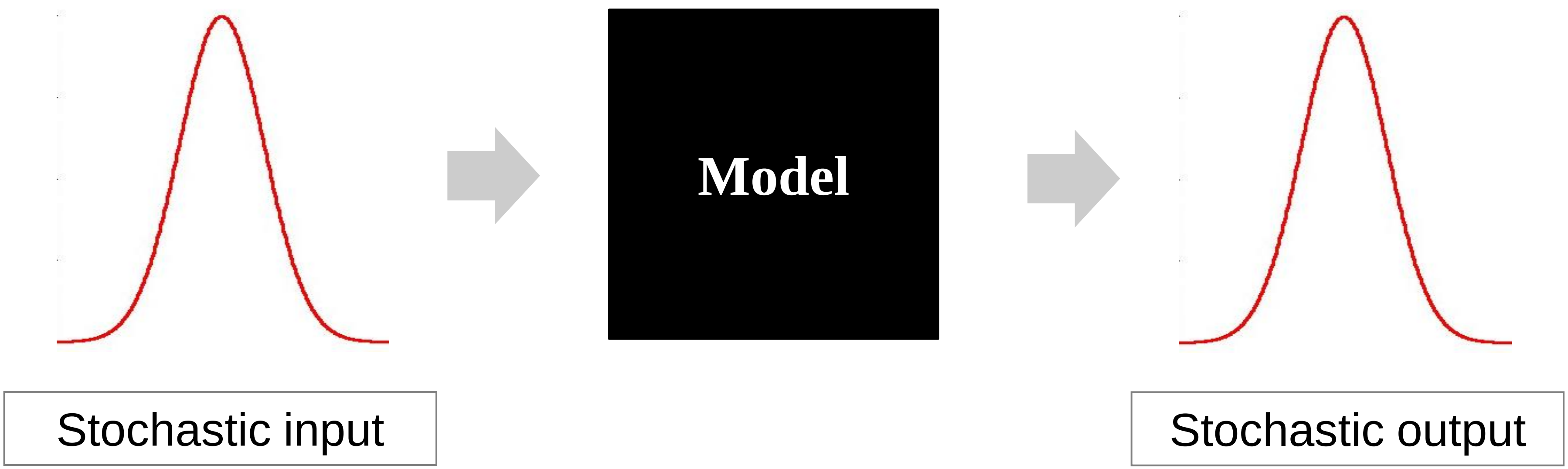
Distributions of random variables



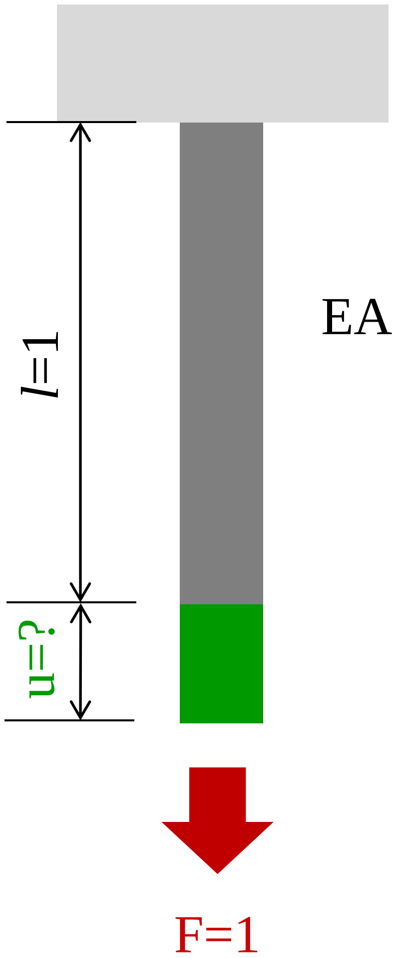
Deterministic model



Stochastic model



Example 1 – Truss element



$$EA \sim \log N(\mu = 1.649, \sigma = 2.16)$$

$$\frac{EA}{l} \cdot U = F$$

$$\log N(\mu = 1.649, \sigma = 2.16) \cdot U = 1$$

$$U \sim \log N(\mu = 1.649, \sigma = 2.16)$$

Problems with a more general model

- output is formulated **implicitly**
- type of **distribution** of the output is **not known**

Idea

- any random variable U can be represented as a series of polynomials in standard normal RVs ξ → $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_M \end{pmatrix}$

$$\mathrm{U} = \sum_{j=0}^{\infty} u_j \cdot \Psi_j(\xi)$$

u_j: coordinates

Ψ_j: polynomial basis

$u_j \cdot \Psi_j(\xi)$: space of all RVs

$\Psi_j(\xi)$ → Hermite Polynomials:
- complete
- orthogonal

Polynomial Chaos

- Representation of a random variable U

$$\mathrm{U} = \sum_{j=0}^{\infty} u_j \cdot \Psi_j(\xi)$$

- Properties

$$\mathrm{E}[g(\xi)] = \int_{-\infty}^{\infty} g(\xi) \cdot \varphi(\xi) d\xi$$

$$\mathrm{E}[\Psi_0] = 1$$

$$\mathrm{E}[\Psi_j] = 0 \qquad \forall j > 0$$

$$\mathrm{E}[\Psi_j \Psi_k] = 0 \qquad \forall j \neq k$$

- Projection of a RV to the Polynomial Chaos

$$u_j = \frac{\mathrm{E}[\mathrm{U} \cdot \Psi_j]}{\mathrm{E}[\Psi_j^2]}$$

Polynomial Chaos approximation

- M random variables
  → M dimensional Hermite Polynomials

- maximum order of the polynomials p

→ P polynomials

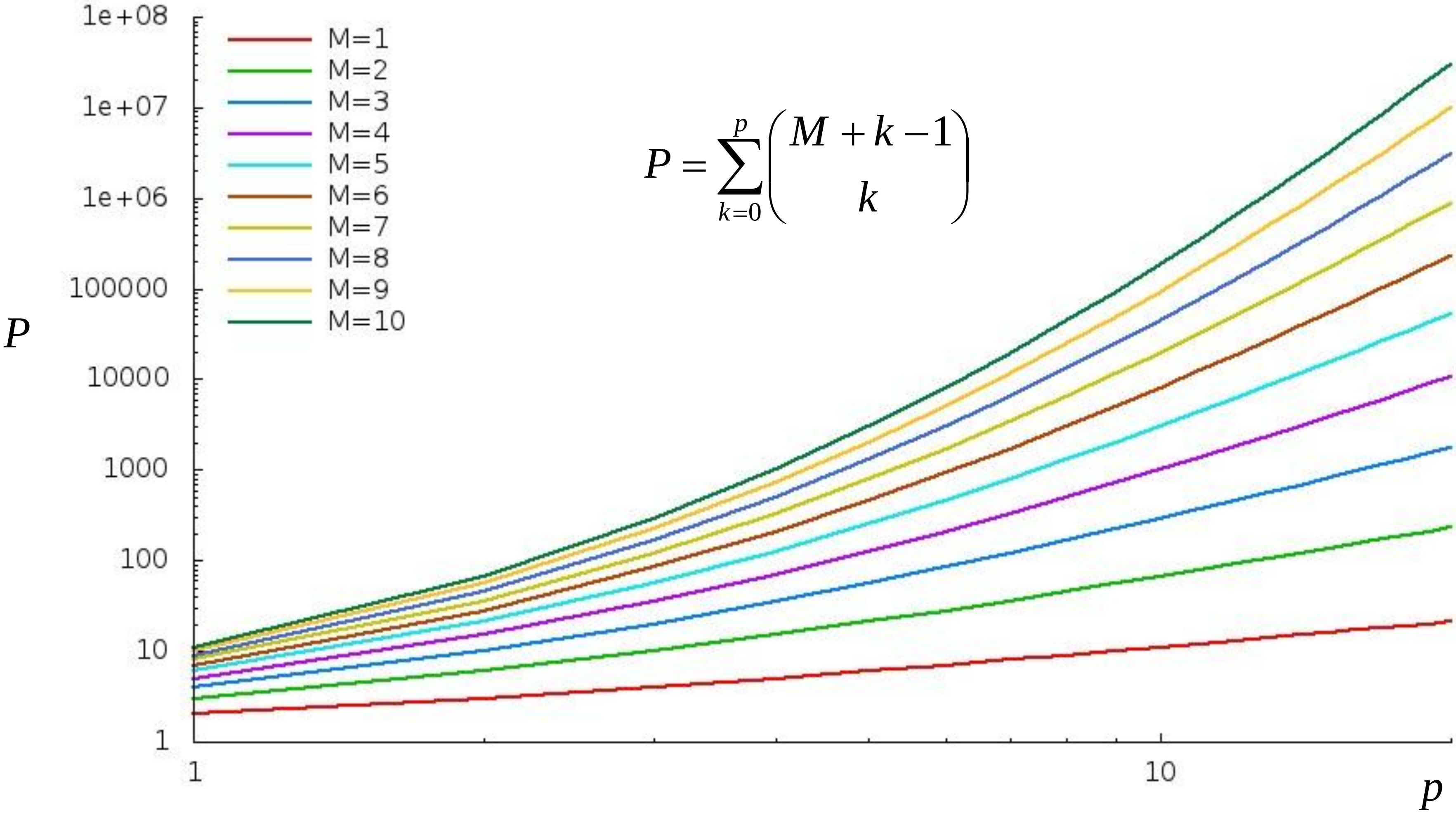
$$P = \sum_{k=0}^{p} \binom{M+k-1}{k}$$

$$\mathrm{U} = \sum_{j=0}^{P-1} u_j \cdot \Psi_j(\xi)$$

$$u_j = \frac{\mathrm{E}[\mathrm{U} \cdot \Psi_j]}{\mathrm{E}[\Psi_j^2]}$$

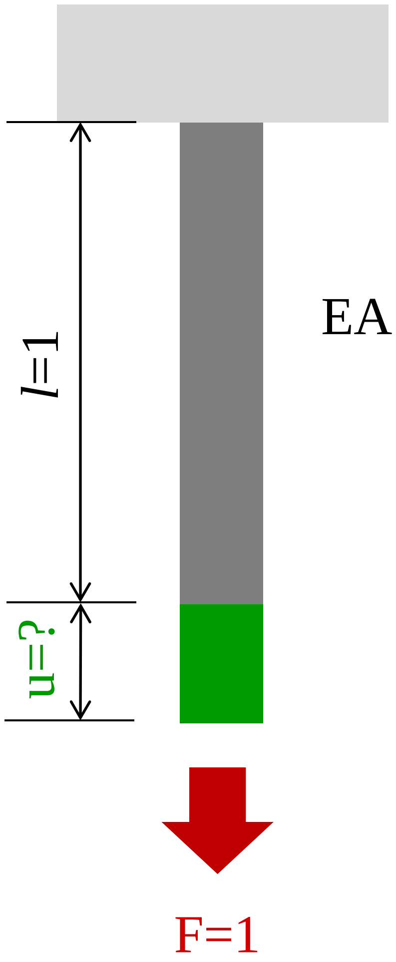
solve this problem

Numer of terms P in the polynomial chaos



$$P = \sum_{k=0}^{p} \binom{M+k-1}{k}$$

Example 1 – Truss element



$$EA \sim \text{logN}(\mu=1.649, \zeta=2.16)$$

Analytical solution:

$$\mathrm{u} = e^{\xi}$$

$$U \sim \text{logN}(\mu = 1.649, \sigma = 2.16)$$

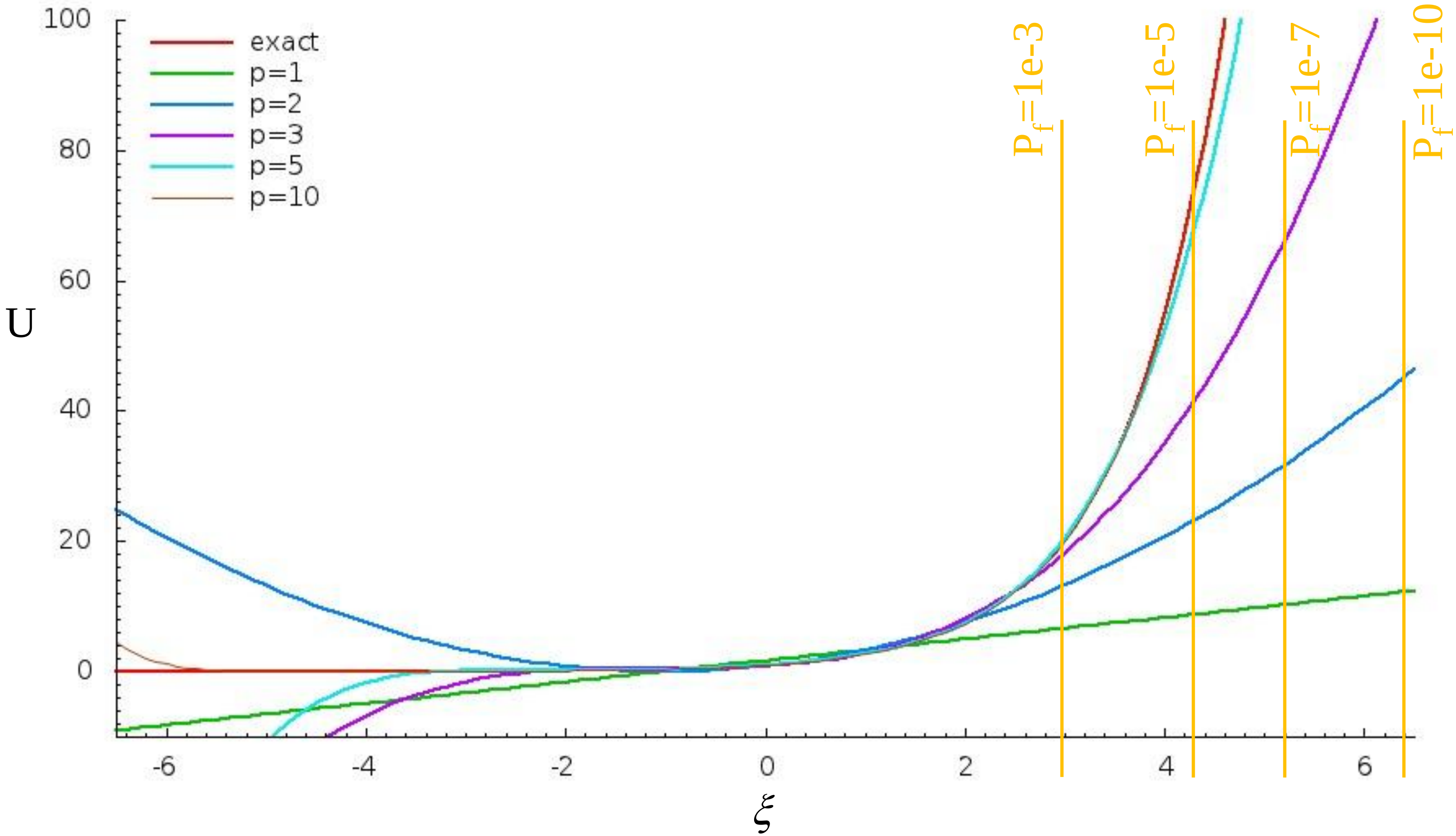
Approximation:

$$\mathrm{U} = \sum_{j=0}^{p} u_j \cdot \Psi_j(\xi)$$

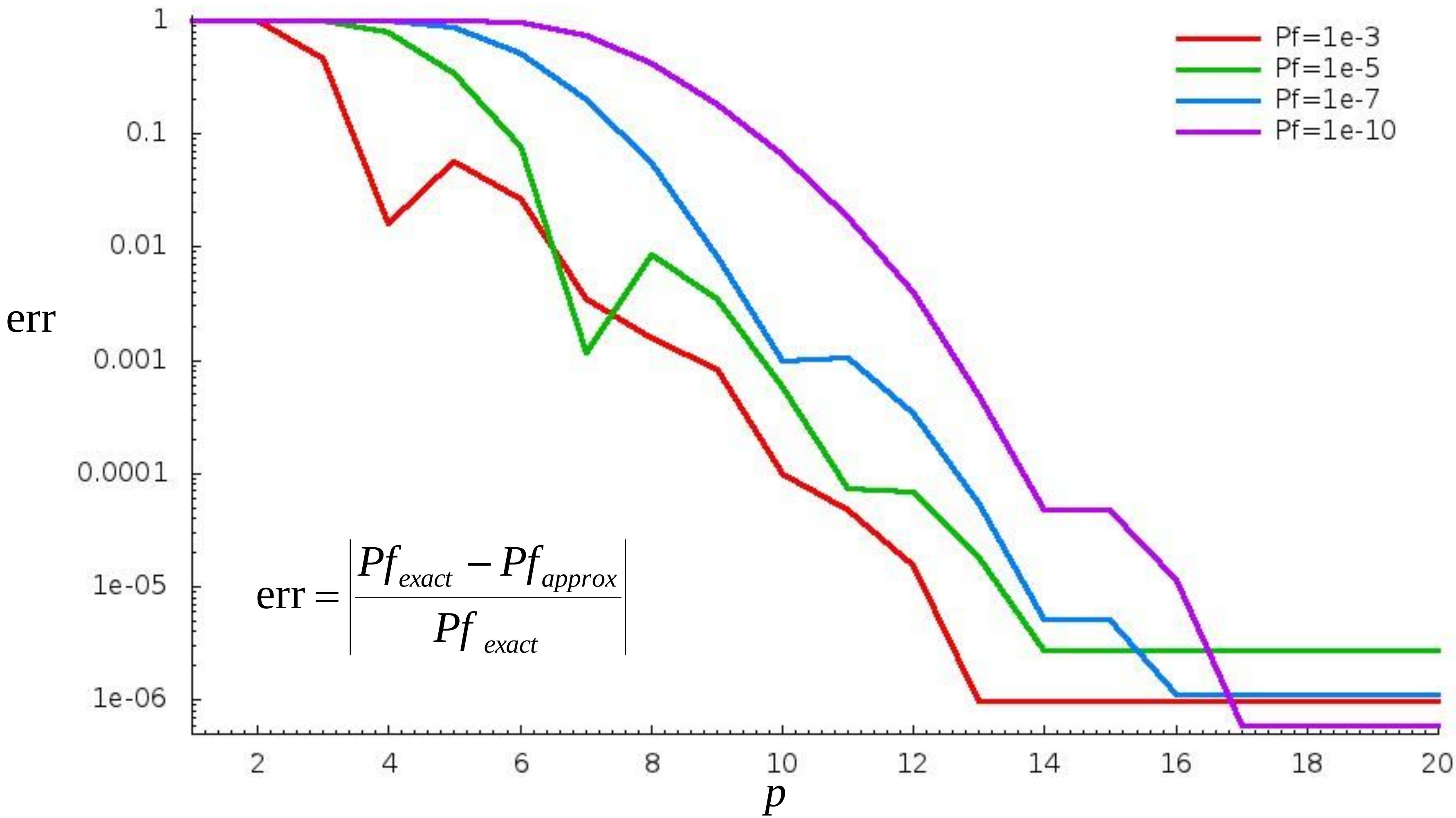
$$u_j = \frac{e^{1/2}}{j!}$$

M=1 → Hermite polynomial basis

Approximation of a Lognormal random variable U

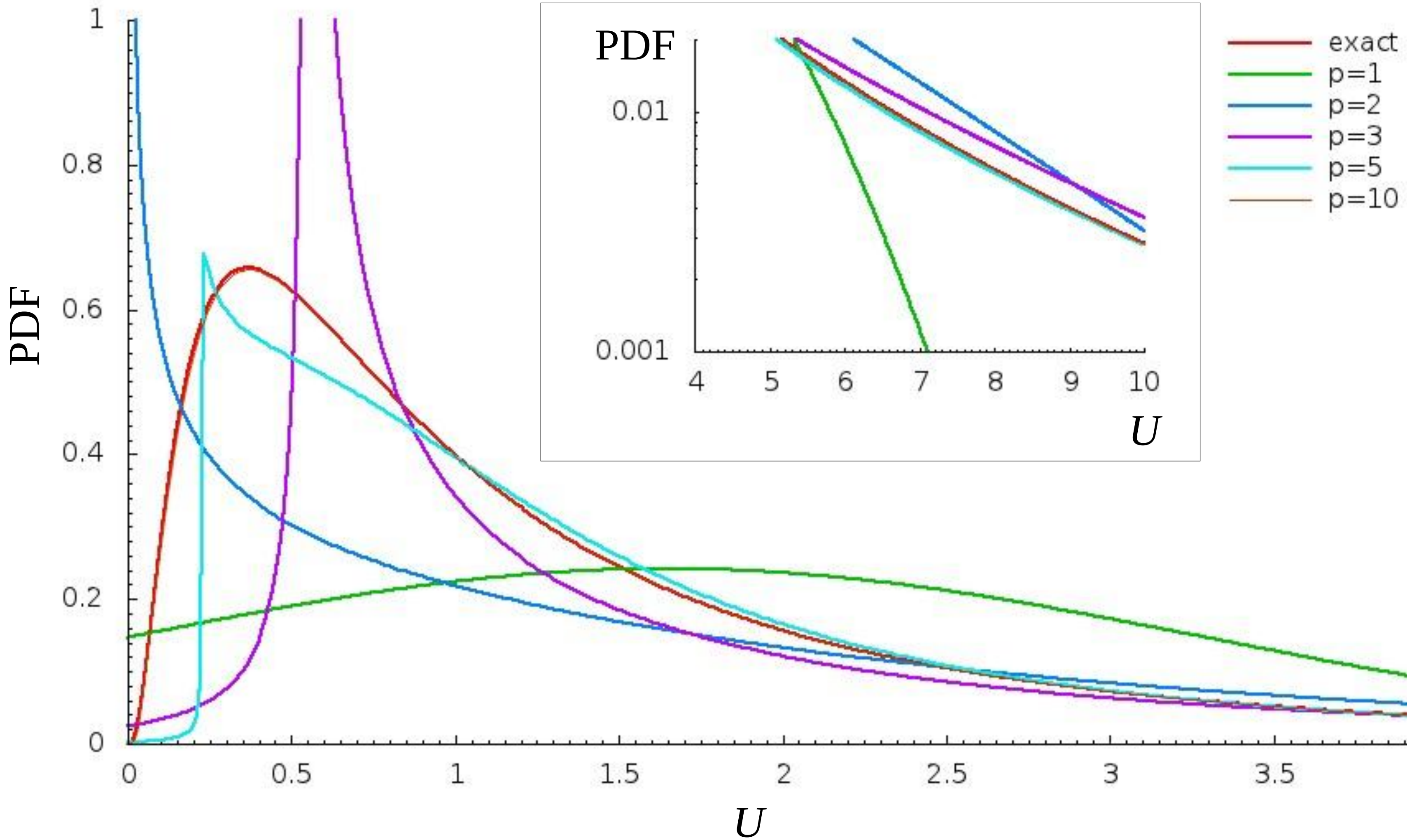


Error of the approximation in the tail of the distribution



$$\mathrm{err} = \left| \frac{Pf_{exact} - Pf_{approx}}{Pf_{exact}} \right|$$

Approximation of the PDF



Computation of the coefficients

- Projection of a RV to the Polynomial Chaos *(numerical integration)*

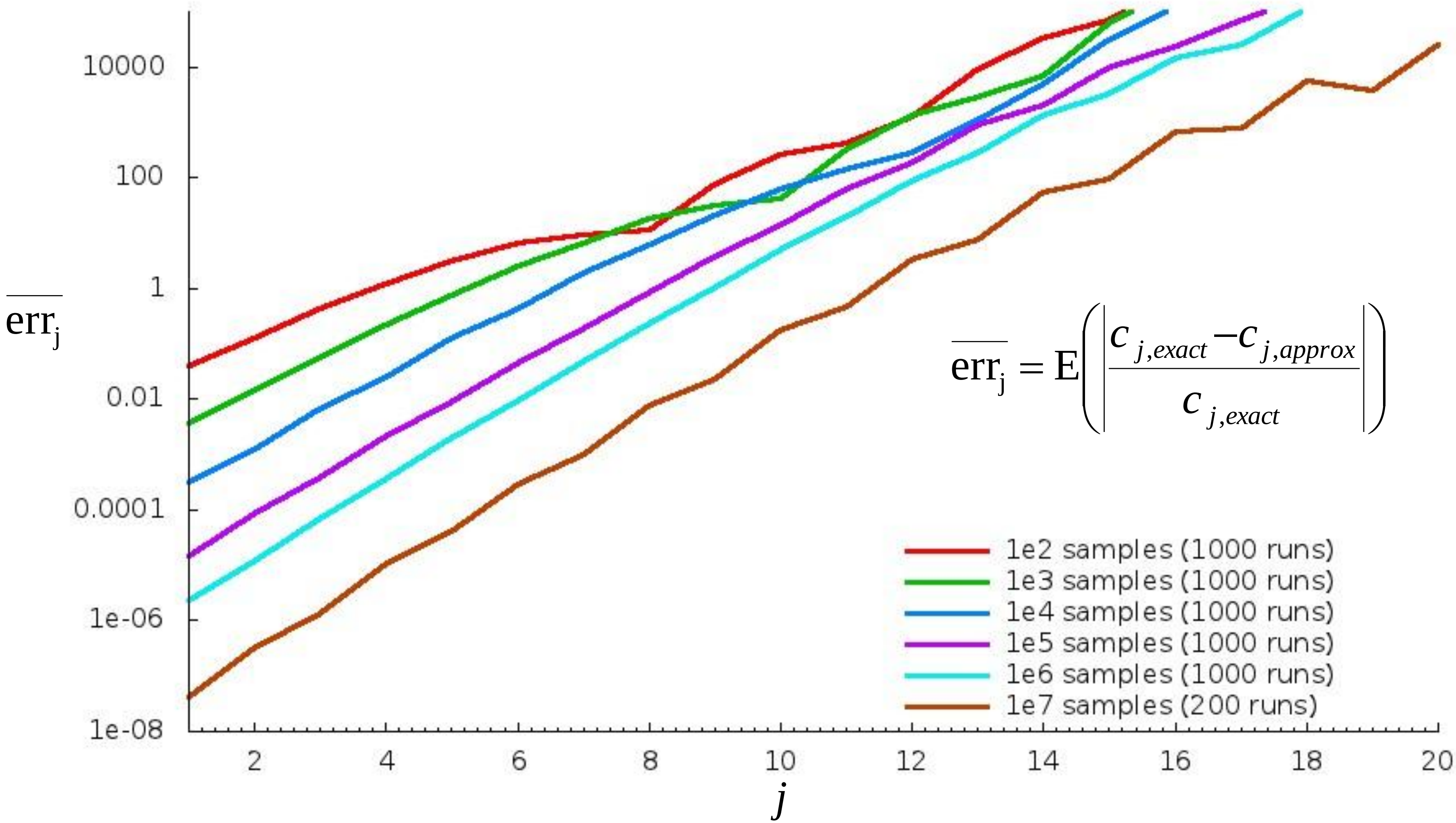
$$u_j = \frac{\mathrm{E}[\mathrm{U}\cdot\Psi_j]}{\mathrm{E}[\Psi_j^2]}$$

→ quite accurate for M=1

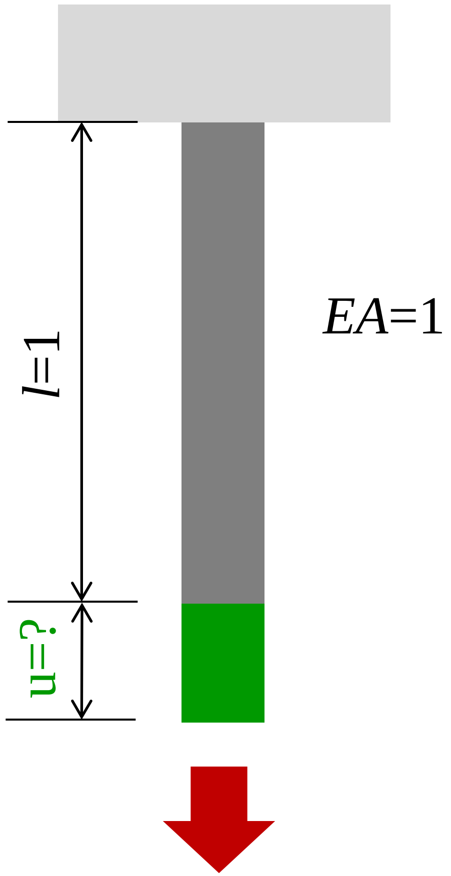
- Linear Regression

$$\hat{\mathbf{Y}} = \arg\min_{\mathbf{Y}\in\mathrm{R}^p} \mathrm{E}\left(\left(\mathrm{U}(\xi) - \mathbf{Y}^{\mathrm{T}}\mathbf{\Psi}(\xi)\right)^2\right)$$

Mean error in the approximation of the coefficients



Example 2 – Truss element



F~Gumbel(u=3;α=0.25)

Analytical solution:

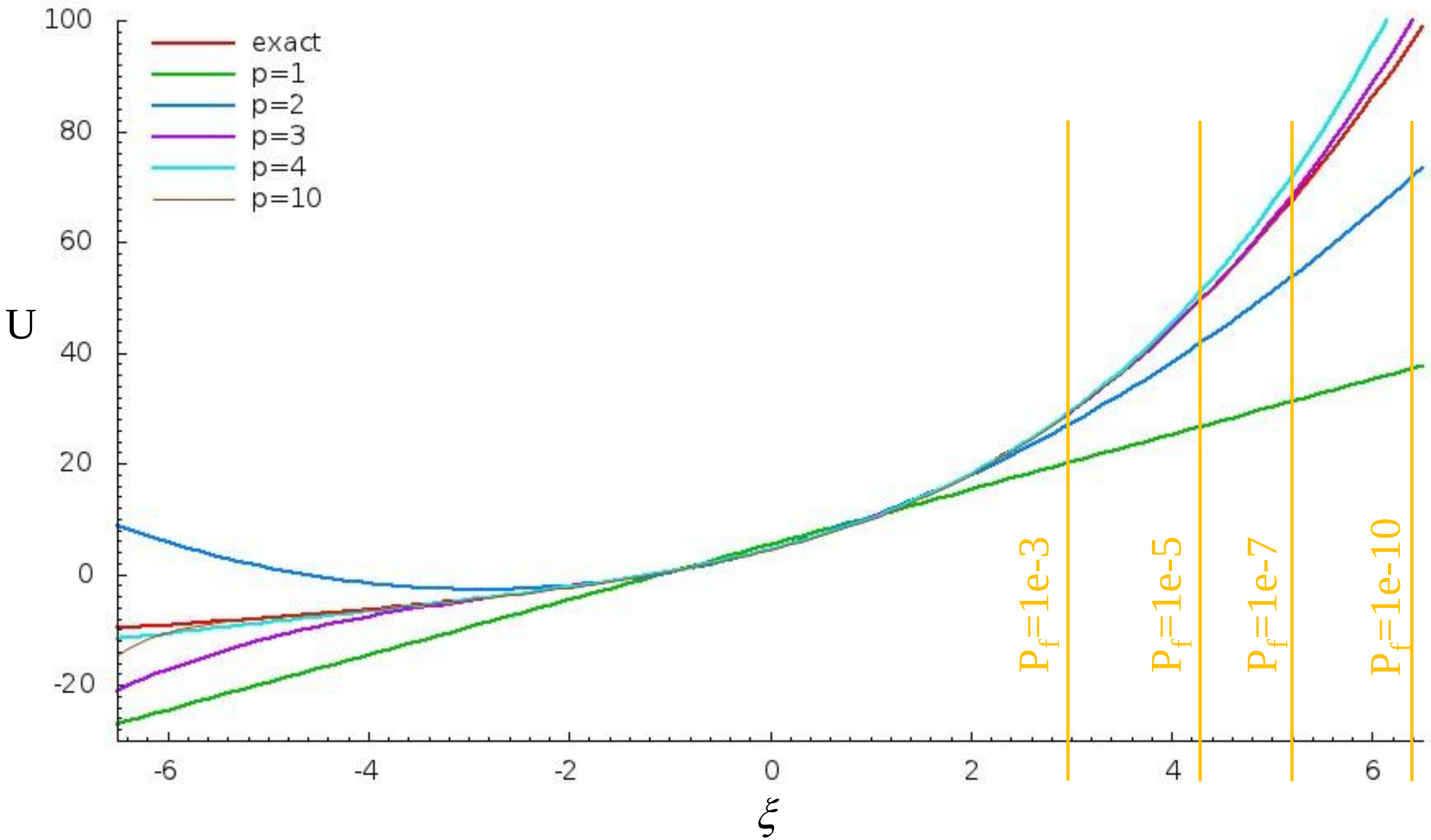
$$U \sim \text{Gumbel}(u=3;\alpha=0.25)$$

Approximation:

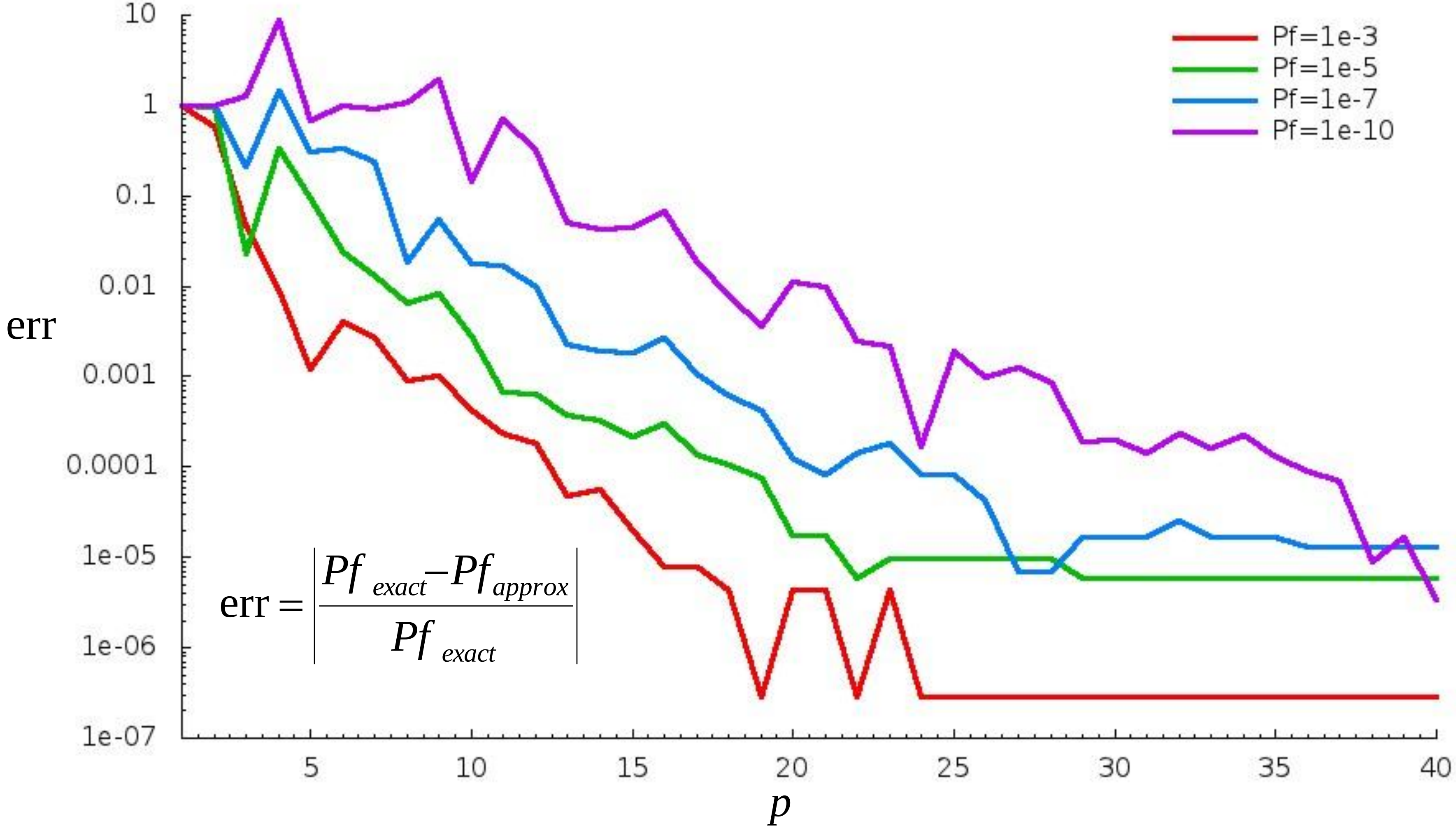
$$U = \sum_{j=0}^{p} u_j \cdot \Psi_j(\xi)$$

$$u_j = \frac{E[U \cdot \Psi_j]}{E[\Psi_j^2]}$$

Approximation of a Gumbel random variable U

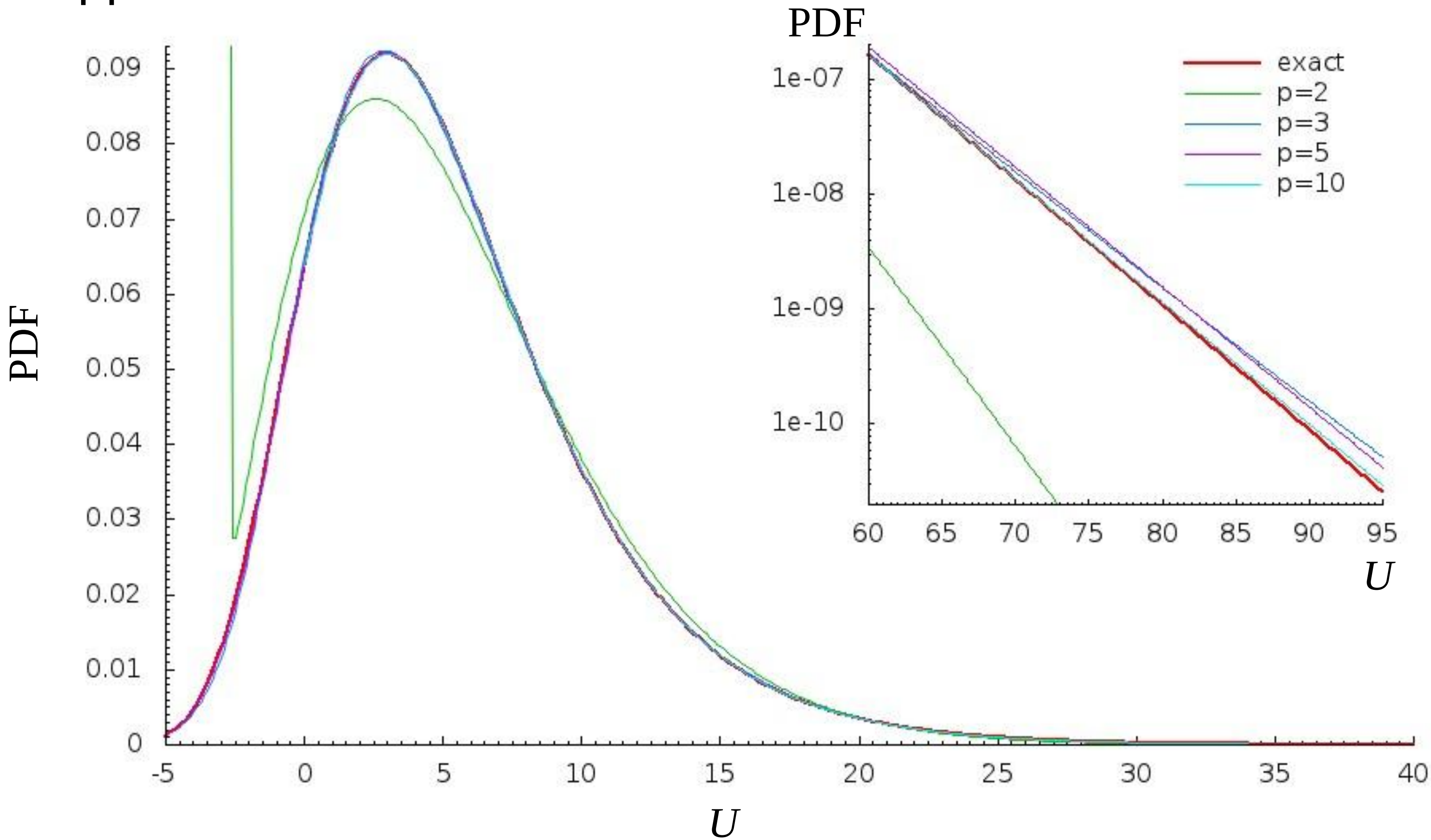


Error of the approximation in the tail of the distribution



$$\text{err} = \left| \frac{Pf_{exact} - Pf_{approx}}{Pf_{exact}} \right|$$

Approximation of the PDF

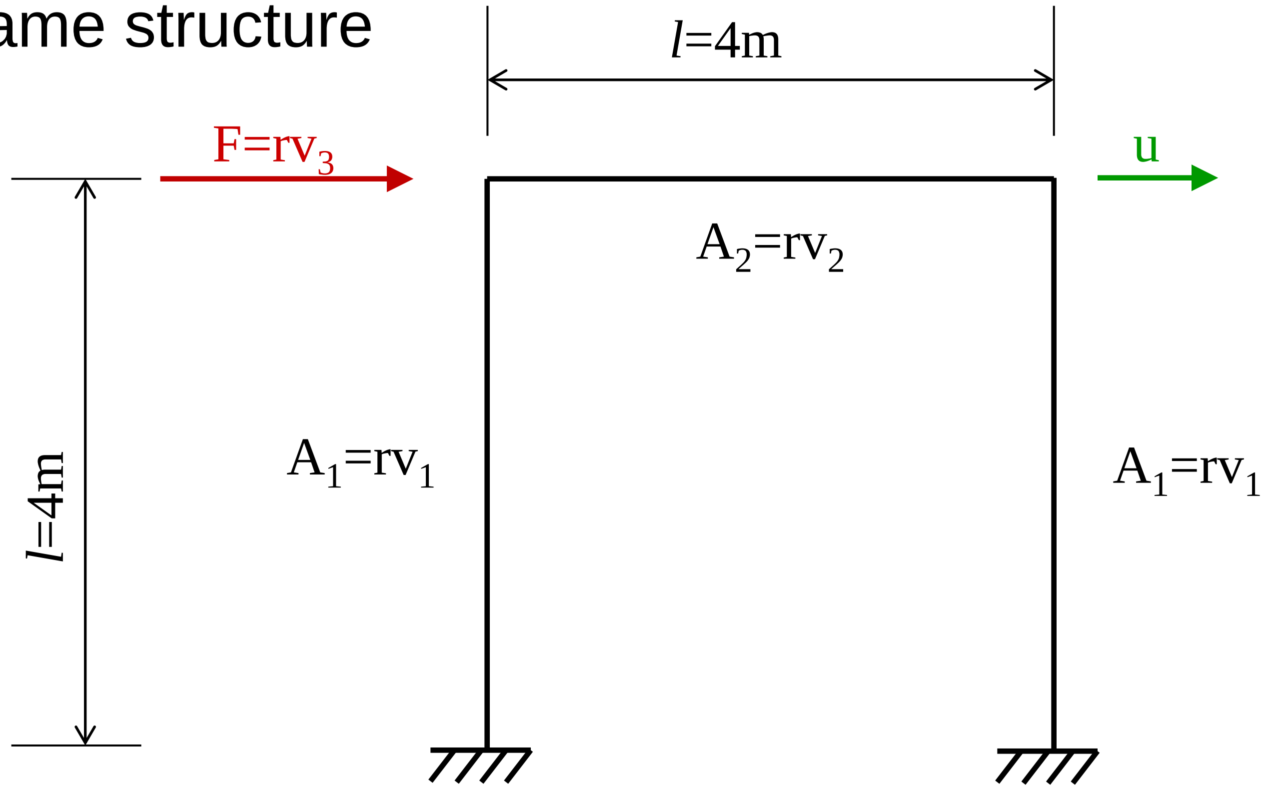


Example 3 – Frame structure

E=2e6kN/m²

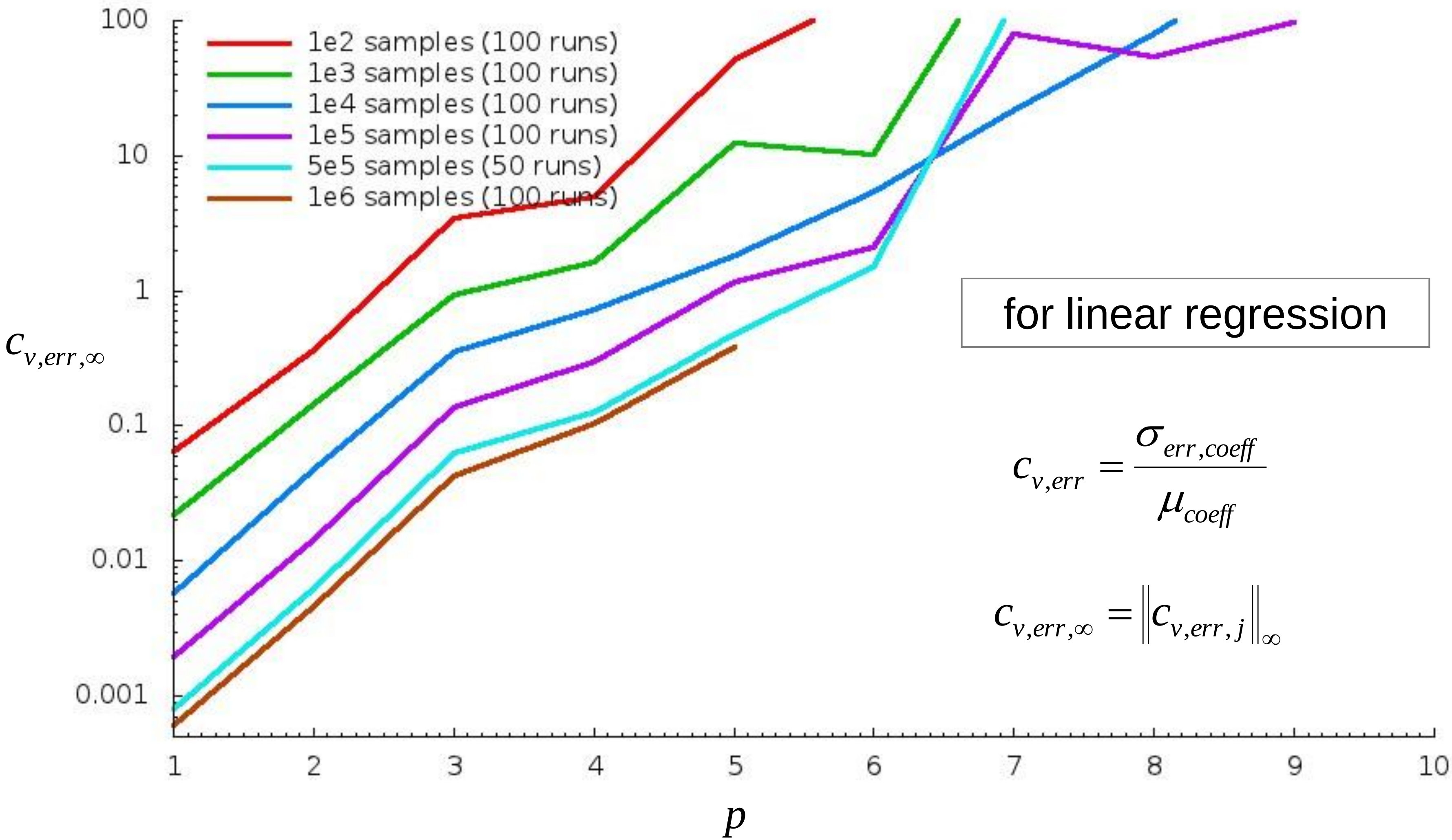
$I_1=0.083333 \cdot rv_1^2$

$I_2=0.16670 \cdot rv_2^2$



Variable	Mean	Standard deviation	Dimension	Distribution
rv1	0.36	0.036	m²	Lognormal
rv2	0.18	0.018	m²	Lognormal
rv3	20	5.0	kN	Gumbel

Coefficient of variation in the approx. of the coefficients

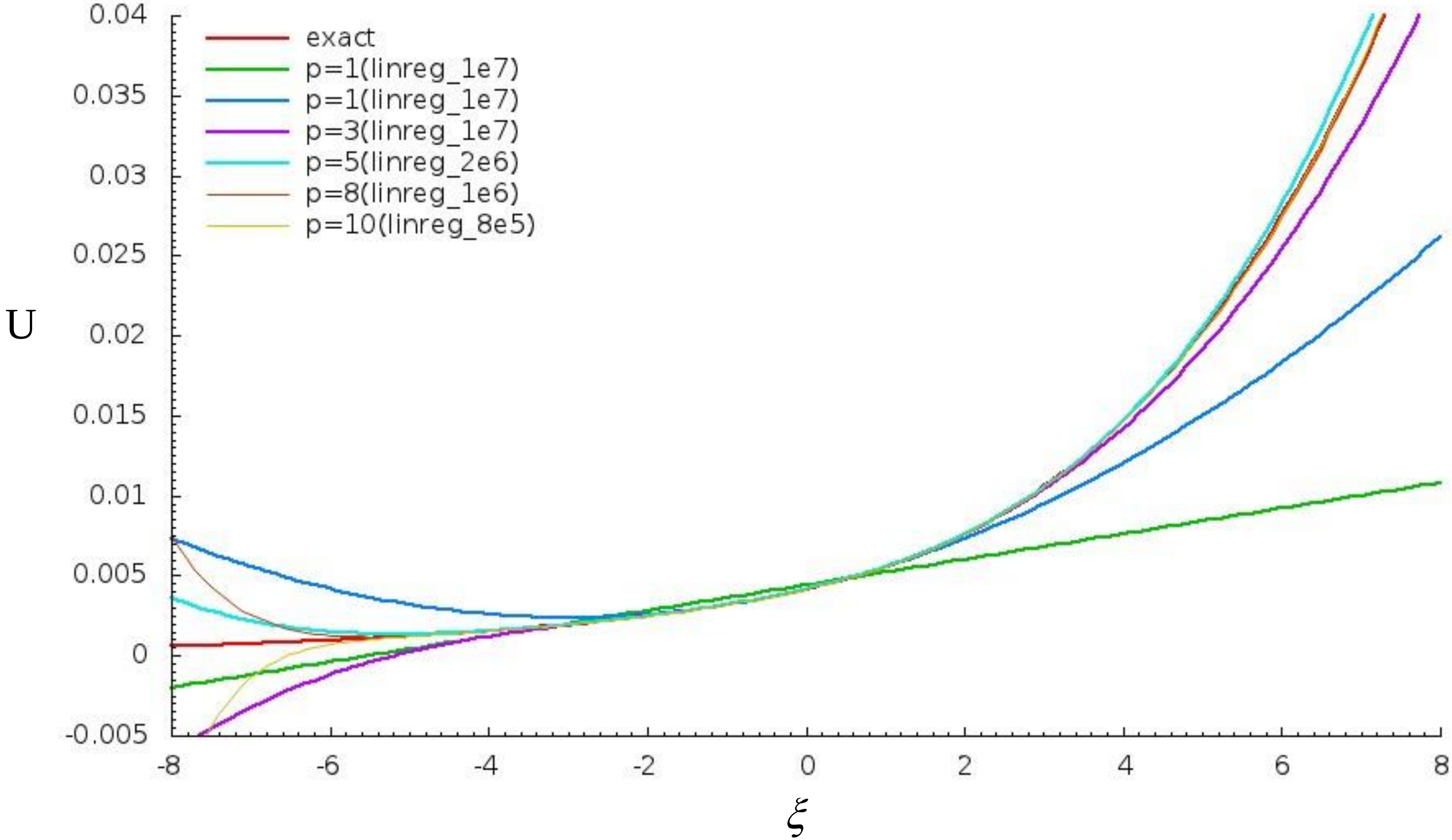


for linear regression

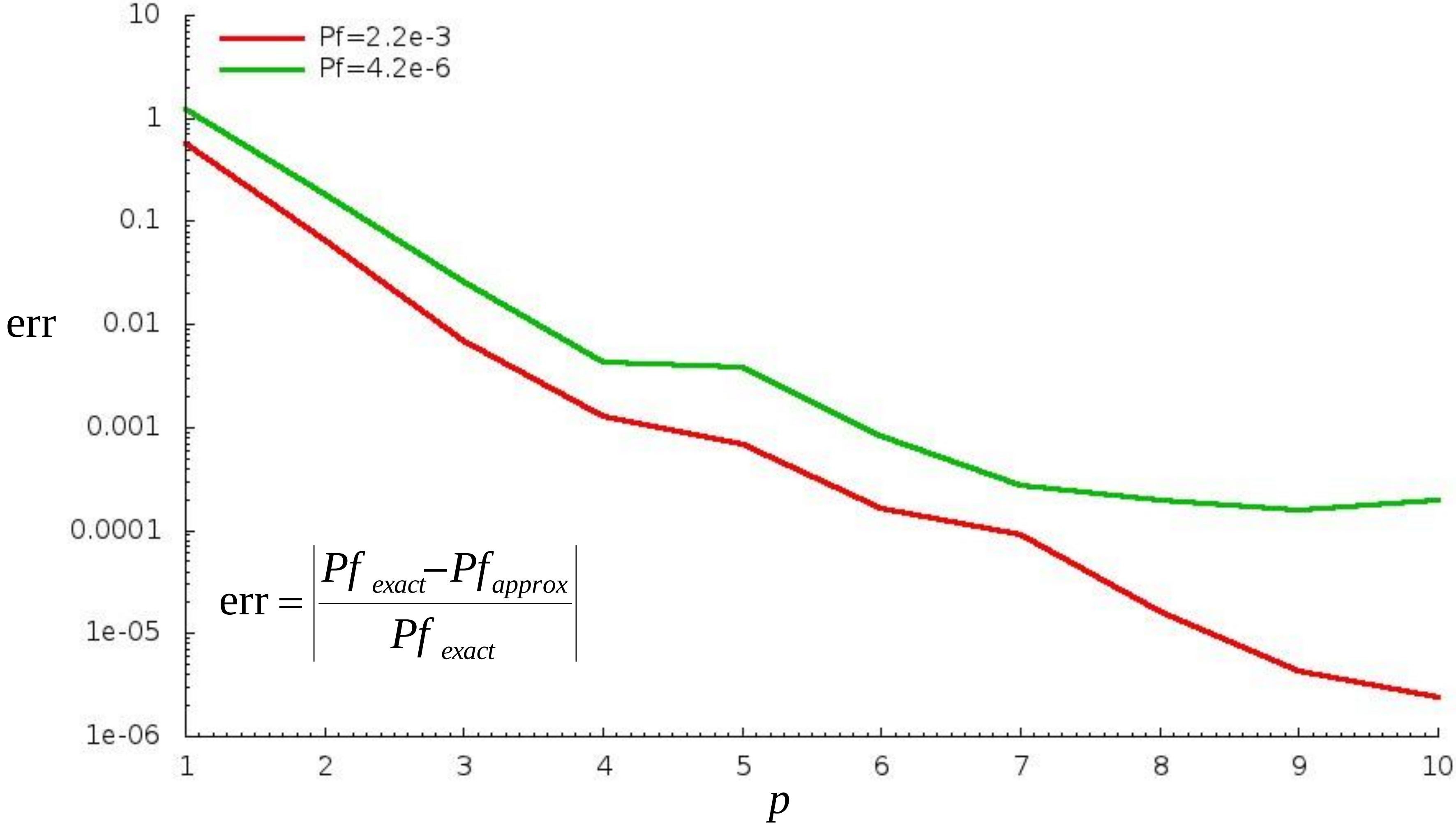
$$c_{v,err} = \frac{\sigma_{err,coeff}}{\mu_{coeff}}$$

$$c_{v,err,\infty} = \left\| c_{v,err,j} \right\|_{\infty}$$

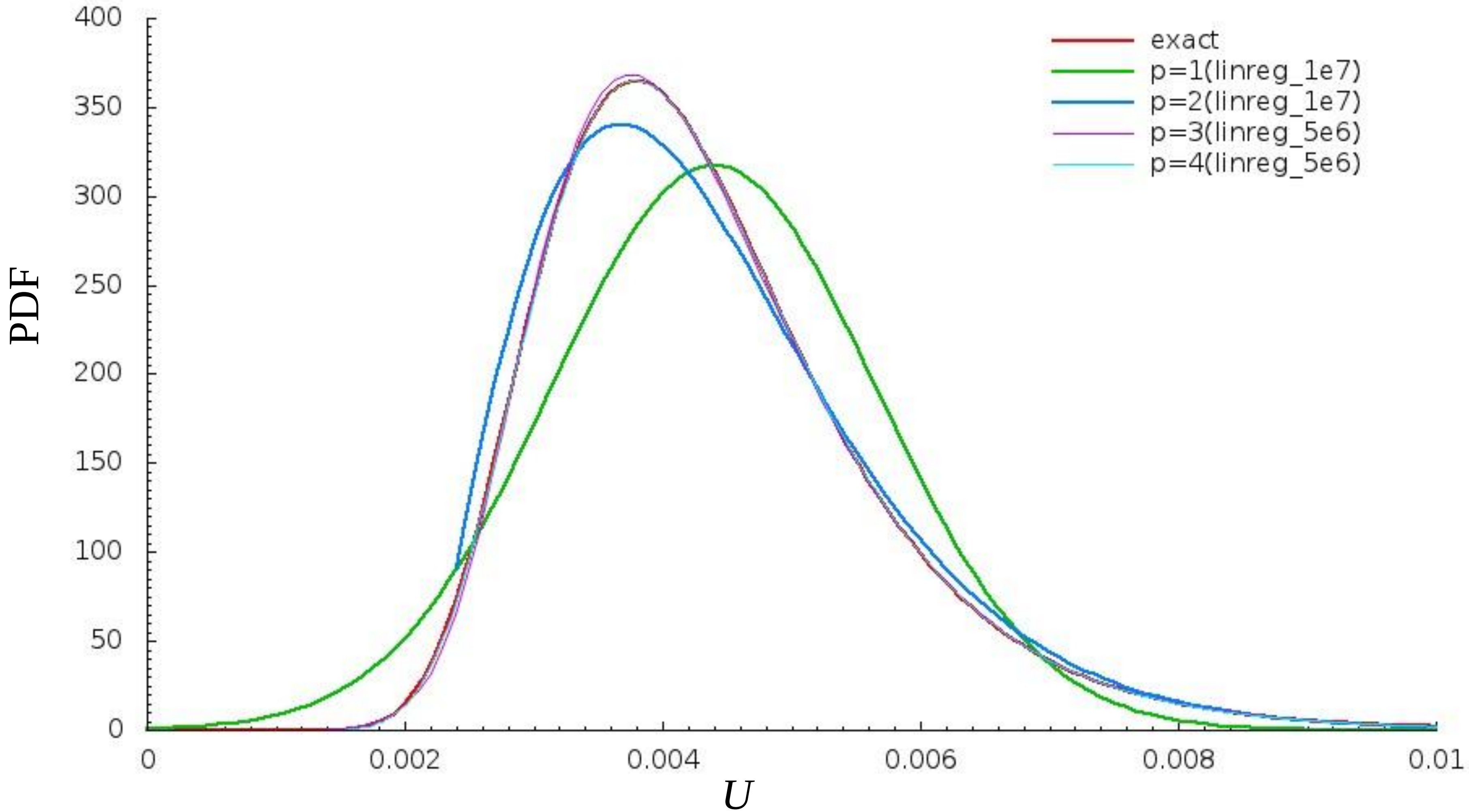
Approx. of the solution U (in direction of the design point)



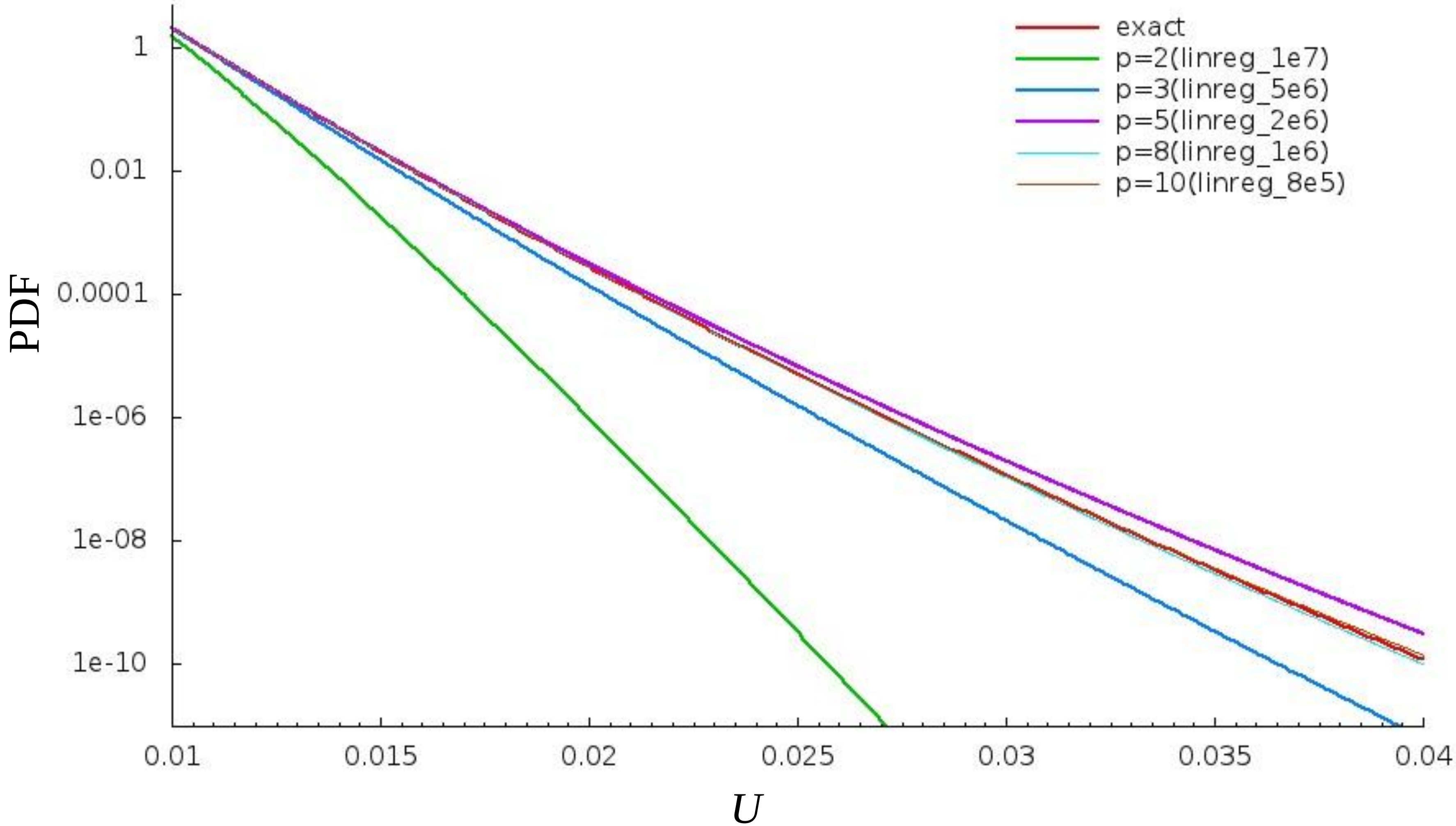
Error of the approximation in the tail of the distribution



Approximation of the PDF (FORM)



Approximation of the tail of the PDF (FORM)



Summary

- Advantages
  - PC is a basis of the space of all random variables
  - Post-processing on an explicit expression
  - Good approx. already for small p close to the mean-values

- Disadvantages
  - For large M: p has to be small
  - Large error in the tail for small p
  - Linear regression and large p → large error in higher c_j's
  - unphysical realization are possible (even for large p)

→ Not the best method for Reliability Analysis (if Pf is small)