Technische Universität München

Bayesian Updating

Development of efficient numerical methods

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Bayesian inference is founded on the famous *Bayes' rule*: $\Pr(A|E)=\Pr(E|A)\cdot\Pr(A)/\Pr(E)$. Bayes's rule states how the probability of an event A changes when new information E becomes available. Bayesian inference can be applied to update our knowledge about parameters that drive a model. In the context of parameters of a model, we typically have to deal with parameters that are continuous (and not discrete). Our initial knowledge about the parameters is formulated in terms of a *prior distribution*. New information that becomes available is expressed in terms of a so-called *likelihood* function. The *posterior distribution* comprises the information contained in the prior and the likelihood, it is proportional to the product of prior and likelihood (*Figure 1*).

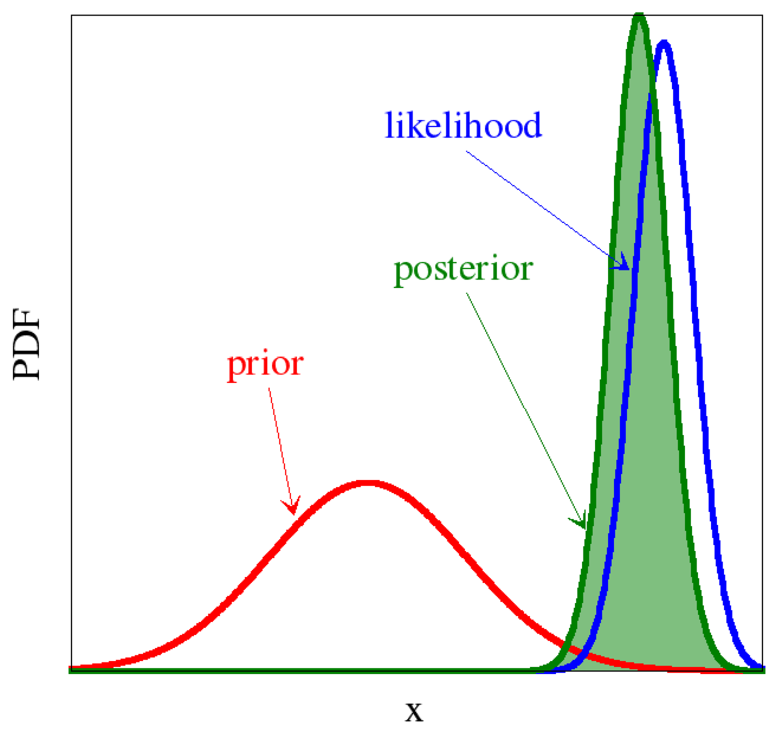

Figure 1. The posterior is proportional to the product of prior and likelihood.

Rejection sampling

Typically, the posterior probabilistic model cannot be derived analytically. Therefore, samples of the posterior distribution have to be generated numerically. A straight-forward approach is to extend the space of uncertain parameters by an additional auxiliary random variable that is uniform between 0 and a value larger or equal than the maximum of the likelihood. We then draw samples from this extended space. A sample whose likelihood is larger or equal than the sample value of the auxiliary variable is a sample from the posterior. This procedure is called rejection sampling and is illustrated in *Figure 2*.

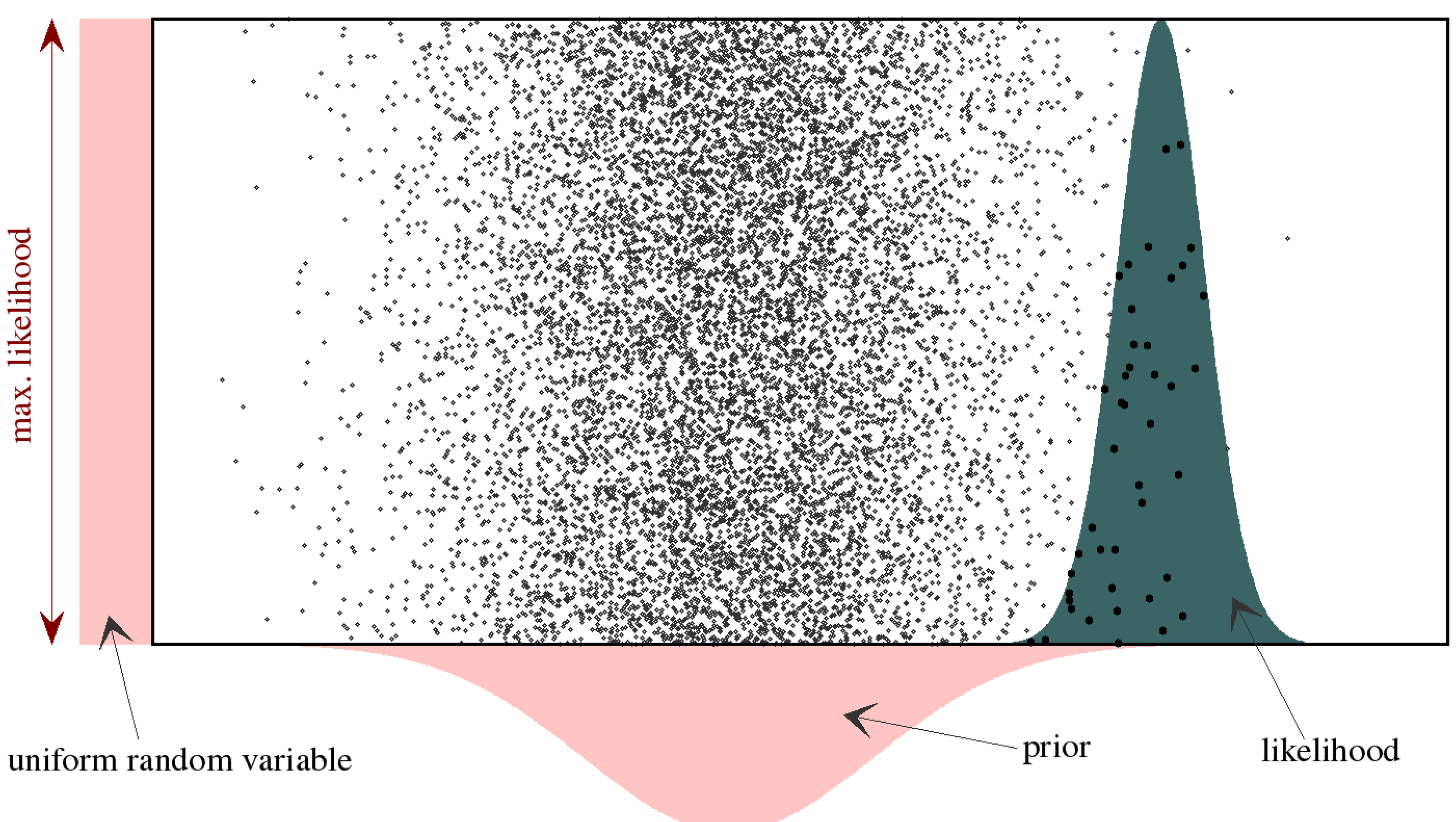

Figure 2. Rejection Sampling is essentially a simple Monte Carlo sampling procedure, where samples that fall below the likelihood are samples from the posterior distribution.

The advantage of rejection sampling is that the generated posterior samples are independent. The main disadvantage is that usually many samples have to be generated to get a single posterior sample. Loosely speaking, the larger the difference between the prior and the likelihood, the more samples are needed. For each sample that is generated the likelihood and, thus, the model has to be evaluated. Therefore, for most models the computational burden of rejection sampling is prohibitive.

The BUS approach

The problem illustrated in *Figure 2* is essentially a problem from structural reliability: The likelihood acts as limit-state function and rejection sampling is equivalent to a crude Monte Carlo sampling. The idea of BUS (Bayesian updating using structural reliability methods) [1] is to solve the problem in *Figure 2* by means of methods that stem from structural reliability. Among all reliability methods, *Subset Simulation* (Au & Beck, 2001) is of particular interest: Subset Simulation is a sampling method for solving structural reliability problems that is particularly efficient for estimating small probabilities in problems with many uncertain variables. The method works by producing samples of a sequence of intermediate distributions conditional on subsets of the failure domain. Conditional samples are obtained through Markov Chain Monte Carlo (MCMC) sampling.

aBUS – adaptive BUS

Both rejection sampling and BUS require the maximum of the likelihood function. If the true maximum is not known, a conservative guess can be made, however, this decreases the efficiency of both approaches. Instead, the procedure of BUS by means of Subset Simulation can be altered such that the maximum of the likelihood function is learned adaptively [2]. This technique is referred to as aBUS.

In aBUS, we introduce an auxiliary random variable that is uniform on the interval $[0,1]$. After each conditioning level in Subset Simulation, we divide the likelihood values by the largest likelihood value observed so far.

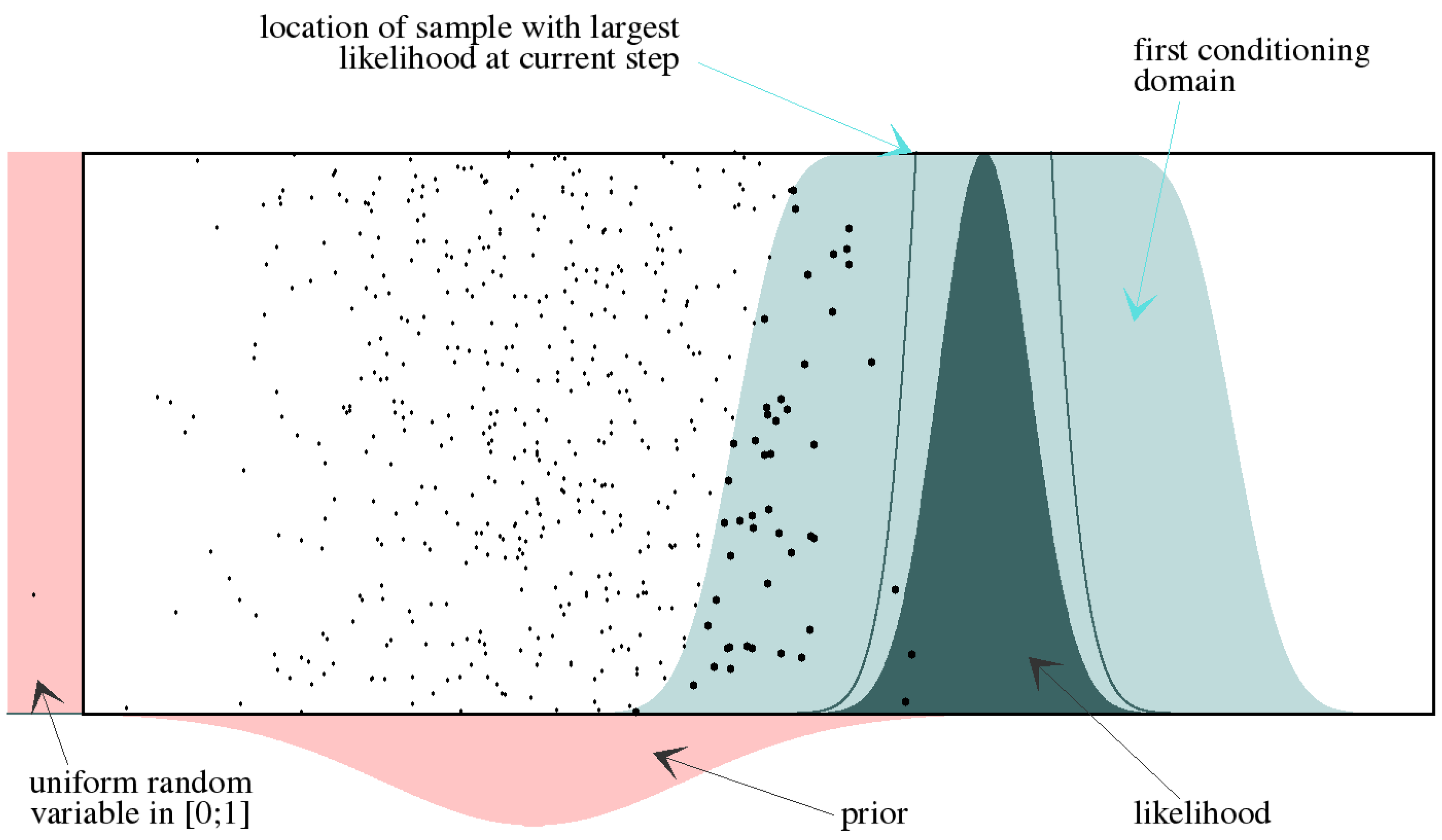

Figure 3. aBUS: initial Monte Carlo sampling and approximate likelihood function for the first conditioning level (based on the sample with the largest likelihood).

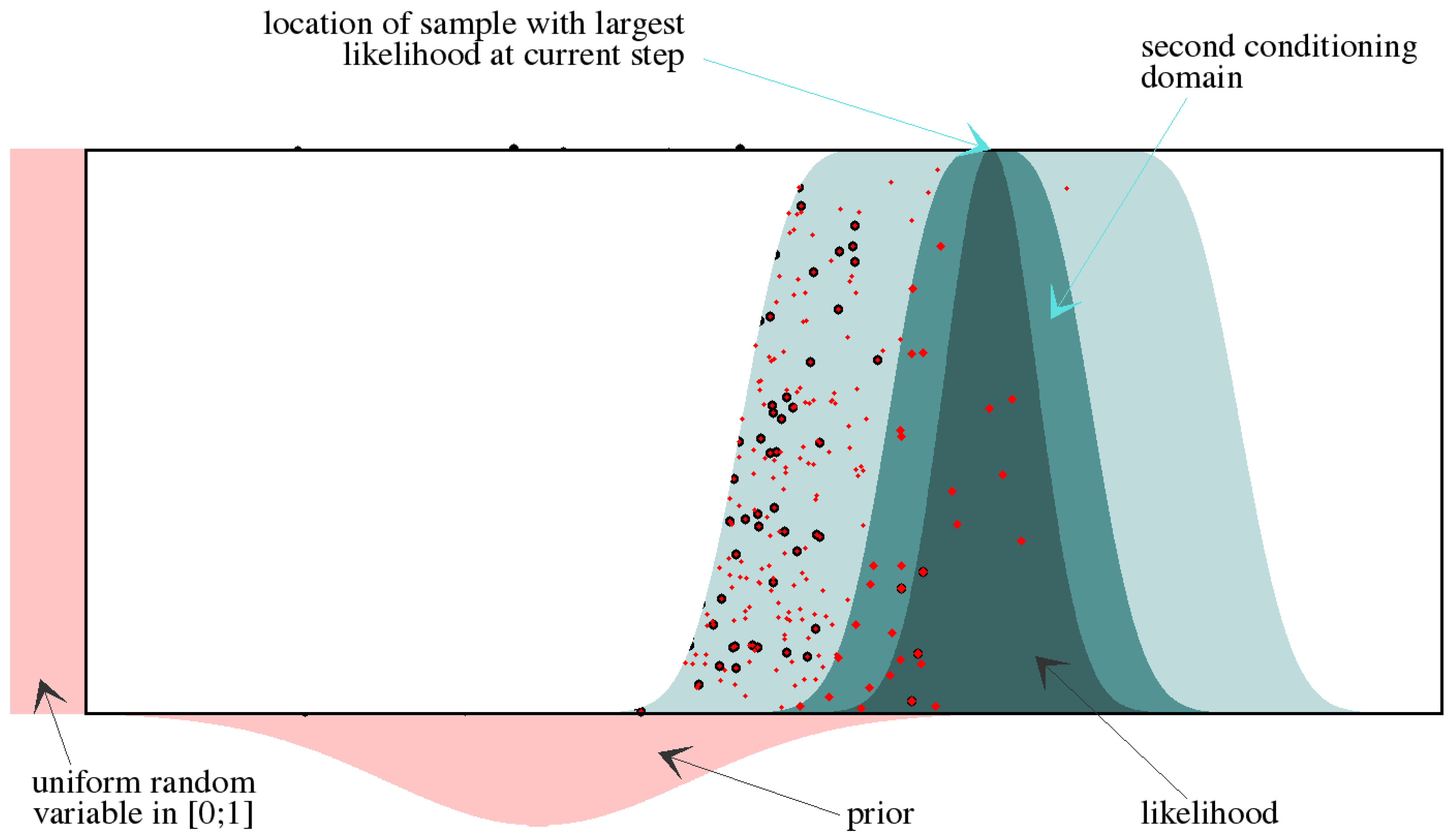

Figure 4. MCMC sampling at the first conditioning level. The second conditioning level is Illustrated. The approximate likelihood in the second conditioning level equals the shape of the true likelihood (because largest observed likelihood equals almost the true maximum).

With increasing levels the samples follow more and more the posterior distribution and the observed maximum approaches the true maximum.

Research interests

Contrary to rejection sampling, BUS and aBUS generate dependent samples. This is due to MCMC sampling that is used to produce samples conditioned on the current intermediate subset (compare *Figure 4*). Loosely speaking, the less efficient the MCMC sampling, the more dependent are the posterior samples. The efficiency of the MCMC sampling depends on an appropriate selection of a so-called proposal distribution. The proposal distribution is described by its type and its spread. How to select and optimize both is one of the questions we try to answer.

Furthermore, we intend to compare BUS and aBUS with other updating procedures, including the TMCMC method (Ching & Chen, 2007) and ABC-SubSim (Chiachio, Beck, Chiachio & Rus, 2014).

Moreover, we are also interested in updating models whose input is driven by a random field. Bayesian inference for such problems is often difficult, because the model typically has many uncertain parameters. However, this should not pose a problem for BUS with Subset Simulation. An interesting question in this context is how to select a prior random field: It is less clear how the uncertainty on the joint probability distribution describing a random field can be consistently represented.

Research code

All algorithms that we develop, investigate and enhance are published in a software package called *Fesslix*. The source code is available free of charge under the Free Software GPL license.

References

[1] D. Straub & I. Papaioannou: *Bayesian updating with structural reliability methods*. Journal of Engineering Mechanics, ASCE, *under Review*.

[2] W. Betz, I. Papaioannou & D. Straub: *Adaptive variant of the BUS approach to Bayesian updating*. 9th International Conference on Structural Dynamics (EURODYN), 2014, Porto, Portugal.